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ANTENNA LABORATORY

Technical Report No. 32

THE EFFICIENCY OF EXCITATION OF A  
SURFACE WAVE ON A DIELECTRIC CYLINDER

by

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## CONTENTS

|                                                                  | <u>Page</u> |
|------------------------------------------------------------------|-------------|
| Acknowledgement                                                  | iii         |
| Abstract                                                         | vii         |
| List of Symbols                                                  | viii        |
| 1. Introduction                                                  | 1           |
| 2. Mathematical Formulation                                      | 5           |
| 3. Solution of the Boundary Value Problem                        | 10          |
| 4. Singularities of the Integrand--Solution of the Mode Equation | 17          |
| 4.1 Branch Points                                                | 17          |
| 4.2 Poles                                                        | 19          |
| 4.3 Solution of the Mode Equation                                | 28          |
| 5. Evaluation of the Contour Integral                            | 30          |
| 5.1 The Radiation Field                                          | 30          |
| 5.2 The Surface Wave Field                                       | 38          |
| 6. The Power Integrals and Excitation Efficiency                 | 42          |
| 6.1 Radiated Power                                               | 43          |
| 6.2 Surface Wave Power                                           | 45          |
| 6.3 Excitation Efficiency                                        | 48          |
| 7. Experimental Investigation                                    | 54          |
| 7.1 The Source Representation and Measurement Method             | 54          |
| 7.2 Experimental Equipment                                       | 58          |
| 7.3 Experimental Results                                         | 66          |
| 7.4 Elimination of the Ground Plane                              | 75          |
| Conclusions                                                      | 76          |
| Bibliography                                                     | 78          |
| Appendix A                                                       | 80          |



## CONTENTS (CONTINUED)

|                   | <u>Page</u> |
|-------------------|-------------|
| Appendix B        | 83          |
| Appendix C        | 106         |
| Distribution List |             |



## ILLUSTRATIONS

| Figure<br>Number                                                                                                              | Page |
|-------------------------------------------------------------------------------------------------------------------------------|------|
| 1. Infinitely Long Dielectric Rod Excited by a Circular Filament<br>of Magnetic Current                                       | 6    |
| 2. Path of Integration in the $\zeta$ Plane                                                                                   | 18   |
| 3. Graph of the Functions $F(X_1) = X_1 \frac{J_0(X_1)}{J_1(X_1)}$ and<br>$F(\xi) = \epsilon_r \xi \frac{K_0(\xi)}{K_1(\xi)}$ | 23   |
| 4. Graph of $X_1 = f(\xi)$ and $X_1^2 + \xi^2 = R^2$                                                                          | 25   |
| 5. Distribution of Poles on the Real Axis                                                                                     | 27   |
| 6. Path of Integration in the $\gamma$ Plane                                                                                  | 32   |
| 7. Circular Cylindrical Coordinate System $(\rho, \phi, z)$ and Spherical<br>Coordinate System $(r, \phi, \theta)$            | 34   |
| 8. Excitation Efficiency as a Function of the Source Dimension<br>$k_0 a$                                                     | 52   |
| 9. Excitation Efficiency as a Function of the Source Dimension<br>$k_0 a$                                                     | 53   |
| 10. View of the Dielectric Rod and Ground Plane                                                                               | 55   |
| 11. Representation of a Two Port Junction                                                                                     | 56   |
| 12. Plot of the Image Circle on the Reflection Coefficient Plane                                                              | 59   |
| 13. Cross Section of the Coaxial Exciter                                                                                      | 61   |
| 14. Coaxial Exciter Mounted on the Ground Plane                                                                               | 64   |
| 15. Diagram of the Measurement Equipment                                                                                      | 65   |
| 16. Measurement of a Junction with a Lossy Short Circuit                                                                      | 67   |
| 17. Arrangement for Measuring Rod Attenuation                                                                                 | 68   |
| 18. Arrangement for Measuring the Transition Efficiency                                                                       | 70   |
| 19. Comparison Between Theoretical and Measured Efficiency                                                                    | 73   |
| 20. Comparison Between Theoretical and Measured Efficiency                                                                    | 74   |
| 21. Solution of the Mode Equation                                                                                             | 83   |



## ABSTRACT

This paper treats the excitation of a transverse magnetic surface wave on a dielectric cylinder. Specifically, the efficiency with which a circular filament of magnetic current excites the  $E_{01}$  mode on an infinite, dielectric rod is determined. The  $E_{01}$  mode is the lowest order, transverse magnetic mode which can propagate on a dielectric rod. The excitation efficiency of a source is defined as the ratio of the power converted to the surface wave mode to the total power which is delivered by the source.

The solution for the field generated by the magnetic current filament is obtained in the form of an inverse Fourier transform. The integral is evaluated by considering it as a contour integral in the plane of the complex propagation constant. The far zone radiation field of the source is obtained from the asymptotic evaluation of the integral by means of the saddle point method of integration. The poles of the integrand yield the TM surface wave modes, which are all circularly symmetric. By restricting the range of  $k_0 b$ , which is the circumference of the dielectric rod in wavelengths, it is assured that only the  $E_{01}$  mode is launched by the source. Curves are presented which show excitation efficiency as a function of  $k_0 a$ , the circumferential length of the filament. It is found that a filament of the proper diameter will excite the  $E_{01}$  mode with an efficiency of approximately 95 percent.

In the laboratory, a very narrow, annular slot in a large metal sheet was used to approximate the magnetic current filament. The excitation efficiency of the slot was measured using Deschamps' method for calibrating a two-port waveguide junction. The measured efficiencies are in close agreement with the results predicted by the theory.



## LIST OF SYMBOLS

|                          |                                                            |
|--------------------------|------------------------------------------------------------|
| $\pi$                    | Constant, $\pi = 3.14159$ .                                |
| $a$                      | Radius of the current filament.                            |
| $b$                      | Radius of the dielectric rod.                              |
| $\epsilon$ or $\epsilon$ | Permittivity.                                              |
| $\mu$                    | Permeability.                                              |
| $\epsilon_r$             | Relative dielectric constant.                              |
| $\sigma$                 | Conductivity                                               |
| $\sigma$                 | Real variable of integration.                              |
| $\omega$                 | Angular frequency.                                         |
| $\lambda_0$              | Free space wavelength.                                     |
| $\lambda_g$              | Guide wavelength.                                          |
| $(\rho, \phi, z)$        | Circular cylinder coordinates.                             |
| $(r, \phi, \theta)$      | Spherical coordinates.                                     |
| $\Delta$                 | Small positive increment.                                  |
| $\delta(z)$              | Dirac delta function.                                      |
| $i$                      | $i = \sqrt{-1}$                                            |
| $k$                      | Wave number, $k = \omega \sqrt{\mu\epsilon}$ .             |
| $\zeta$                  | Complex propagation constant.                              |
| $\nu_0$                  | $\nu_0 = \sqrt{k_0^2 - \zeta^2}$ .                         |
| $\nu_1$                  | $\nu_1 = \sqrt{k_1^2 - \zeta^2}$ .                         |
| $X_0$                    | $X_0 = \nu_0 b$ .                                          |
| $X_1$                    | $X_1 = \nu_1 b$ .                                          |
| $\xi$                    | $\xi = X_0/i$ .                                            |
| $R$                      | $R = \sqrt{X_1^2 + \xi^2} = k_0 b \sqrt{\epsilon_r - 1}$ . |



|                |                                                          |
|----------------|----------------------------------------------------------|
| $J_n(v)$       | Bessel function, first kind, order $n$ .                 |
| $Y_n(v)$       | Bessel function, second kind, order $n$ .                |
| $H_n^{(1)}(v)$ | Hankel function, first kind, order $n$ .                 |
| $H_n^{(2)}(v)$ | Hankel function, second kind, order $n$ .                |
| $I_n(v)$       | Modified Bessel function, first kind, order $n$ .        |
| $K_n(v)$       | Modified Bessel function, second kind, order $n$ .       |
| $h(\rho, \xi)$ | Fourier transform of $H_\phi(\rho, z)$ .                 |
| $\vec{K}$      | Vector magnetic current density, volts per square meter. |
| $\alpha$       | Dielectric attenuation constant.                         |
| $\gamma$       | $\gamma = \epsilon_r \xi \frac{K_0(\xi)}{K_1(\xi)}$      |
| $\Gamma$       | Reflection coefficient                                   |
| $\tau$         | Complex variable, $\tau = \psi + i\eta$ .                |
| $w(\tau)$      | $w(\tau) = \sqrt{\epsilon_r \sin^2 \tau}$                |
| $W^I$          | Surface wave power in region I.                          |
| $W^{II}$       | Surface wave power in region II.                         |
| $W^S$          | Total surface wave power, $W^S = W^I + W^{II}$ .         |
| $W^R$          | Radiated power.                                          |
| $\Upsilon$     | Excitation efficiency.                                   |



## 1. INTRODUCTION

A number of papers published in the last decade have treated the mode characteristics of surface waves on various types of open waveguides. Of particular interest has been the utilization of such structures as surface wave antennas. A factor of prime importance in all surface guide applications is the efficient excitation of the desired mode on the guide. The excitation efficiency of a source is defined as the ratio of the power converted to the surface wave mode to the total power which is delivered by the source.

Most of the published articles on surface wave excitation efficiency have treated sources placed near infinite planar structures such as the dielectric coated metal sheet. In order to determine the excitation efficiency of a source, one must solve the source form of Maxwell's equations which amounts to solving an inhomogeneous wave equation. The usual technique of solving the differential equation is to apply the method of integral transforms which yields the solution for the field in the form of a definite integral. The integral is evaluated by considering it as a contour integral in the plane of the complex propagation constant. The integrand has poles and branch points in the complex plane. By deforming the contour of integration in a suitable manner, the integral becomes equal to the sum of the residues at the poles plus a branch cut integral. The poles correspond to the surface wave modes and the amplitude of a surface mode is given by the residue of the integrand at the pole. The branch cut integration yields the radiation field. Alternatively, the radiation field is obtained from



the asymptotic evaluation of the integral by means of a saddle point integration. This method of analysis yields the amplitude of the radiation field and of the surface mode. One may then calculate the radiated power and surface wave power and determine the excitation efficiency of the source.

Several authors have applied this method to particular problems. Cullen<sup>1</sup> calculated the efficiency of an infinitely long slot above a dielectric-coated plane conductor. He also treated the slot above a corrugated surface and the case of a chopped surface wave source such as a gently flared horn. His theory predicted efficiencies greater than 90 percent for the infinite slot above the corrugated surface. Whitmer<sup>2</sup> determined the efficiency of an infinitely long, electric current filament placed in a dielectric sheet. The maximum efficiency calculated for this source was approximately 85 percent. The problems treated by Cullen and Whitmer are primarily of academic interest, since they involve infinite source distributions and doubly infinite planar structures. As yet, the efficiencies predicted in Cullen's and Whitmer's work have not been verified experimentally.

Roberts<sup>3,4</sup> investigated the single wire transmission line excited from a flanged coaxial cable. This source launches the transverse magnetic,  $E_{00}$  mode on the wire. He evaluated the input conductance of the coaxial line by assuming an infinitesimal gap between the wire and the outer conductor of the coax. He defines a radiation conductance  $G_1$  and a surface mode conductance  $G_0$ . The excitation efficiency of the source may be obtained from  $G_1$  and  $G_0$ . Roberts measured the properties of two different wires in the laboratory. Part of his experimental



investigation included measuring the input conductance of the coaxial line for a number of different gap radii. The measured conductances were plotted on a graph and the curve was extrapolated to obtain the conductance for an infinitesimal gap, since it was impossible to measure such a source experimentally. The extrapolated value agreed quite closely with the conductance predicted from theory. To the author's knowledge, this is the only previous experimental corroboration of efficiency as predicted from the pole, branch cut analysis.

Since only limited experimental verification of this method of analysis has appeared in the literature, this research was directed toward obtaining more complete experimental confirmation for a practical theoretical problem. The surface waveguide which was chosen for the problem is the dielectric rod since it is a surface wave structure which has practical application. The source is a circular filament of magnetic current placed inside the dielectric rod concentric with the longitudinal axis of the rod. The magnetic current ring was selected since it is a source which has a simple physical realization. One would expect the ring to be an efficient exciter of the lowest order, circularly symmetric, transverse magnetic mode which can propagate on a dielectric rod. This mode is known as the  $E_{01}$  mode. The purpose of this investigation is to determine the efficiency with which the magnetic current ring excites the  $E_{01}$  mode using the pole, saddle point integration method, and then to measure the efficiency experimentally. It will be seen later that excitation efficiency may be readily measured using Deschamps<sup>5,6</sup> method for determining the scattering matrix coefficients of a two port waveguide junction.



A similar problem has been treated by C. Jauquet,<sup>7</sup> who selected a magnetic current ring which was greater in diameter than the dielectric rod. In his paper, Jauquet obtains an integral solution for the field but does not carry out the evaluation of the integral, nor does he present any calculations of excitation efficiency. His experimental measurements were concerned with the phase velocity and radial distribution of the surface wave mode.



## 2. MATHEMATICAL FORMULATION

The problem to be investigated is illustrated in Fig. 1. An infinitely long, dielectric rod of radius  $b$  is located such that its longitudinal axis corresponds to the  $z$  axis of circular cylinder coordinates  $(\rho, \phi, z)$ . The rod is considered lossless,  $\sigma = 0$ , with a magnetic permeability  $\mu_0$  and permittivity  $\epsilon_1 = \epsilon_r \epsilon_0$ , where  $\epsilon_r$  is the relative dielectric constant. The medium surrounding the rod and extending to infinity is free space, with constants  $\mu_0$  and  $\epsilon_0$ . The electromagnetic field source is a filamentary ring of magnetic current located at the plane  $z = 0$ . The ring is of radius  $a$ , where  $0 < a < b$ , and is infinitesimally small in cross section. The source distribution is represented as a product of Dirac delta functions in the  $\rho$  and  $z$  coordinates as follows:

$$\bar{K} = \bar{\phi} \delta(\rho-a) \delta(z) \quad (2.1)$$

where  $\bar{\phi}$  is a unit vector in the  $\phi$  direction. The source distribution  $\bar{K}$  is independent of  $\phi$  (uniform), and is a unit source such that

$$\iint \bar{K} \cdot \bar{\phi} da = \int_{a-\Delta}^{a+\Delta} \int_{0-\Delta}^{0+\Delta} \delta(\rho-a) \delta(z) \cdot dz d\rho = 1$$

since

$$\int_{a-\Delta}^{a+\Delta} \delta(\rho-a) d\rho = 1, \text{ and } \int_{0-\Delta}^{0+\Delta} \delta(z) dz = 1.$$

$\bar{K}$  has the dimensions of volts per square meter. A harmonic time dependence of  $e^{-i\omega t}$  is selected for the source.



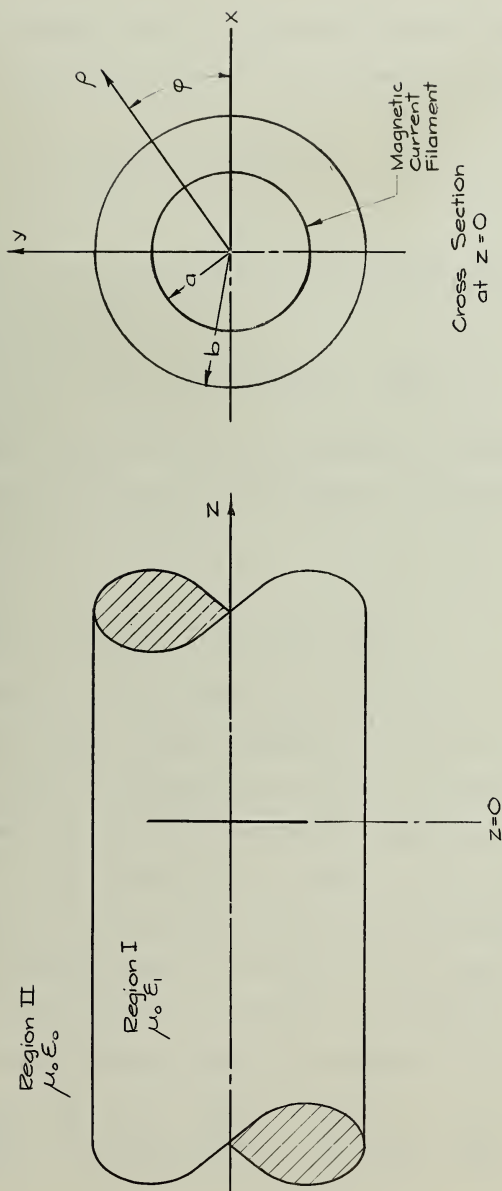


FIGURE 1. INFINITELY LONG DIELECTRIC ROD EXCITED BY A CIRCULAR FILAMENT OF MAGNETIC CURRENT



The electromagnetic field is a solution of Maxwell's equations.

Written in differential form for  $e^{-i\omega t}$  time dependence, we have

$$\begin{aligned}\nabla \times \bar{\mathbf{E}} &= + i\omega \mu \bar{\mathbf{H}} - \bar{\mathbf{K}} \\ \nabla \times \bar{\mathbf{H}} &= - i\omega \epsilon \bar{\mathbf{E}}.\end{aligned}\tag{2.2}$$

Taking the curl of the second equation and then substituting the first relation for  $\nabla \times \bar{\mathbf{E}}$  yields

$$-\nabla \times \nabla \times \bar{\mathbf{H}} + \omega^2 \mu \epsilon \bar{\mathbf{H}} = -i\omega \epsilon \bar{\mathbf{K}}.\tag{2.3}$$

The only non-zero component of  $\bar{\mathbf{K}}$  is the coefficient of the unit vector  $\bar{\phi}$ . We may write the  $\bar{\phi}$  component of the vector eq. (2.3) and obtain the non-homogenous scalar equation

$$[-\nabla \times \nabla \times \bar{\mathbf{H}}]_{\phi} + \omega^2 \mu \epsilon H_{\phi} = -i\omega \epsilon \delta(\rho-a) \delta(z),$$

where the  $\bar{\phi}$  component of the bracketed term is indicated.

The magnetic current filament generates a field having components  $H_{\phi}$ ,  $E_{\rho}$ , and  $E_z$ , while the components  $E_{\phi}$ ,  $H_{\rho}$ , and  $H_z$  are equal to zero. Due to the symmetry of the source, the field is independent of  $\phi$  and we note that all partial derivatives with respect to  $\phi$  must be zero. It is evident that the source produces a field which is circularly symmetric and transverse magnetic with respect to the  $z$  axis. Expanding the bracketed term in cylindrical coordinates, we obtain the partial differential equation relating  $H_{\phi}(\rho, z)$  to the source

$$\frac{\partial^2 H_{\phi}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_{\phi}}{\partial \rho} + (k^2 - \frac{1}{\rho^2}) H_{\phi} + \frac{\partial^2 H_{\phi}}{\partial z^2} = -i\omega \epsilon \delta(\rho-a) \delta(z)\tag{2.4}$$



where  $k^2 = \omega^2 \mu \epsilon$ .

From  $\bar{E} = -\frac{1}{i\omega\epsilon} \nabla \times \bar{H}$ , it follows that

$$\begin{aligned} E_\rho(\rho, z) &= \frac{1}{i\omega\epsilon} \frac{\partial H_\phi}{\partial z} \\ E_z(\rho, z) &= \frac{1}{-i\omega\epsilon} \left( \frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi \right) \end{aligned} \quad (2.5)$$

In order to solve (2.4) for  $H_\phi$ , we shall apply the method of integral transforms to reduce (2.4) to a non-homogeneous, ordinary differential equation. We define the Fourier transform of  $H_\phi(\rho, z)$  as<sup>8</sup>

$$h(\rho, \zeta) = \int_{-\infty}^{+\infty} H_\phi(\rho, z) e^{-i\zeta z} dz. \quad (2.6)$$

The inverse transform is given by

$$H_\phi(\rho, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(\rho, \zeta) e^{+i\zeta z} d\zeta. \quad (2.7)$$

Assuming that the transform of  $H_\phi(\rho, z)$  exists, we multiply each term of (2.4) by  $e^{-i\zeta z}$  and integrate over the infinite range with respect to  $z$  to obtain

$$\frac{d^2 h}{d\rho^2} + \frac{1}{\rho} \frac{dh}{d\rho} + \left( k^2 - \zeta^2 - \frac{1}{\rho^2} \right) h = -i\omega\epsilon \delta(\rho-a) \quad (2.8)$$

where the source variation  $\delta(z)$  is no longer present since

$$\int_{-\infty}^{+\infty} \delta(z) e^{-i\zeta z} dz = 1.$$



We may consider (2.8) as an ordinary differential equation in which  $\zeta$  is a parameter constant. The boundary conditions on  $h(\rho, \zeta)$  are obtained by taking the Fourier transform of the original boundary conditions on  $H_\phi(\rho, z)$  and  $\frac{\partial H_\phi}{\partial \rho}$ . The problem is one of solving (2.8) for  $h(\rho, \zeta)$  subject to the transformed boundary conditions.  $H_\phi(\rho, z)$  is then obtained by use of the inversion integral (2.7). The integral expression for  $H_\phi(\rho, z)$  is derived in the following chapter.



## 3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

Consider the form of the non-homogeneous differential equation (2.8). One could solve (2.8) using the Hankel transform; however, the definition of the delta function allows one to solve (2.8) in a simpler manner. The delta function is defined by the relations

$$\int_{a-\Delta}^{a+\Delta} \delta(\rho-a) d\rho = 1, \text{ and } \delta(\rho-a) = 0 \text{ for } \rho \neq a.$$

Consequently, for all  $\rho$  other than  $\rho = a$ , equation (2.8) reduces to the homogeneous differential equation

$$\frac{d^2 h}{d\rho^2} + \frac{1}{\rho} \frac{dh}{d\rho} + (k^2 - \zeta^2 - \frac{1}{\rho^2}) h = 0 \quad (3.1)$$

The delta function  $\delta(\rho-a)$  implies a boundary condition which  $h(\rho, \zeta)$  must satisfy at  $\rho = a$ . Multiplying each term of (2.8) by  $d\rho$  and integrating over the interval  $2\Delta$  from  $\rho = a - \Delta$  to  $\rho = a + \Delta$ , one obtains

$$\begin{aligned} \left[ \frac{dh}{d\rho} \right]_{a-\Delta}^{a+\Delta} + \int_{a-\Delta}^{a+\Delta} \frac{1}{\rho} \left( \frac{dh}{d\rho} \right) d\rho + (k^2 - \zeta^2) \int_{a-\Delta}^{a+\Delta} h d\rho + \\ + \frac{1}{\rho} h \left[ \right]_{a-\Delta}^{a+\Delta} - \int_{a-\Delta}^{a+\Delta} \frac{1}{\rho} \left( \frac{dh}{d\rho} \right) d\rho = -i\omega \varepsilon \int_{a-\Delta}^{a+\Delta} \delta(\rho-a) d\rho \end{aligned} \quad (3.2)$$

Assuming that  $h(\rho, \zeta)$  is continuous for all  $\rho$ , in the limit as  $\Delta \rightarrow 0$  and  $\rho \rightarrow a$ , Eq. (3.2) reduces to

$$\left[ \frac{dh}{d\rho} \right]_{\rho=a} = -i\omega \varepsilon \quad (3.3)$$



We have shown that a continuous  $h(\rho, \zeta)$  which satisfies the homogeneous equation (3.1) and whose first derivative is discontinuous by  $-i\omega\epsilon$  at  $\rho = a$  is a solution of (2.8).

The remaining boundary conditions on  $h(\rho, \zeta)$  follow from the boundary conditions imposed on  $H_\phi(\rho, z)$  and  $E_z(\rho, z)$  by Maxwell's equations. Referring to Fig. 1, we denote the cross sectional area of the rod as region I and the space outside the rod as region II. Since tangential  $\vec{H}$  is continuous at a magnetic current discontinuity, we see that  $H_\phi(\rho, z)$  is continuous at  $\rho = a$  for all  $z$  including the filament position  $z = 0$ . Since tangential  $\vec{E}$  and  $\vec{H}$  are continuous across a dielectric boundary, we note that  $H_\phi(\rho, z)$  and  $E_z(\rho, z)$  must be continuous at  $\rho = b$  for all  $z$ . The corresponding conditions on  $h(\rho, \zeta)$  are:

- (i) Since  $H_\phi(\rho, z)$  is continuous at  $\rho = a$  for all  $z$ ,

$$h(\rho, \zeta) = \int_{-\infty}^{+\infty} H_\phi(\rho, z) e^{-i\zeta z} dz$$

must also be continuous at  $\rho = a$ , hence

$$h(\rho, \zeta)]_{a-\Delta} = h(\rho, \zeta)]_{a+\Delta} \quad (3.4)$$

- (ii) Similarly, since  $H_\phi(\rho, z)$  is continuous at  $\rho = b$  for all  $z$ ,

$$h(\rho, \zeta)]_{b-\Delta} = h(\rho, \zeta)]_{b+\Delta} \quad (3.5)$$

- (iii) From (2.5) we write

$$E_z(\rho, z) = \frac{1}{-i\omega\epsilon} \left( \frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi \right).$$



In region I, where  $0 \leq \rho \leq b$ , the permittivity  $\epsilon_1 = \epsilon_r \epsilon_0$ . In region II, where  $\rho > b$ ,  $\epsilon = \epsilon_0$ .  $E_z(\rho, z)$  continuous at  $\rho = b$  for all  $z$  requires

$$\left[ \frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi \right]_{b-\Delta} = \epsilon_r \left[ \frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi \right]_{b+\Delta}$$

Taking the transform, we obtain

$$\left[ \frac{dh}{d\rho} + \frac{1}{\rho} h \right]_{b-\Delta} = \epsilon_r \left[ \frac{dh}{d\rho} + \frac{1}{\rho} h \right]_{b+\Delta}$$

which reduces to

$$\epsilon_r \left[ \frac{dh}{d\rho} \right]_{b+\Delta} - \left[ \frac{dh}{d\rho} \right]_{b-\Delta} + \left[ \frac{1}{\rho} h \right]_{b+\Delta} (\epsilon_r - 1) = 0 \quad (3.6)$$

by reason of (3.5)

In order to determine  $H_\phi(\rho, z)$  in region II, we must solve Eq. (2.8) for the corresponding  $h(\rho, \zeta)$ . Inside the dielectric rod, for all  $\rho$  except  $\rho = a$ , (2.8) becomes

$$\frac{d^2 h}{d\rho^2} + \frac{1}{\rho} \frac{dh}{d\rho} + (k_1^2 - \zeta^2 - \frac{1}{\rho^2}) h = 0 \quad (3.7)$$

where  $k_1^2 = \omega^2 \mu_0 \epsilon_1$ .

Outside the dielectric rod and for all  $\rho$ , (2.8) becomes

$$\frac{d^2 h}{d\rho^2} + \frac{1}{\rho} \frac{dh}{d\rho} + (k_0^2 - \zeta^2 - \frac{1}{\rho^2}) h = 0 \quad (3.8)$$

where  $k_0^2 = \omega^2 \mu_0 \epsilon_0$ .



We proceed by writing general solutions  $h(\rho, \zeta)$  of (3.7) for the regions  $0 \leq \rho \leq a$ , and  $a \leq \rho \leq b$ . The discontinuity at  $\rho = a$  appears in the boundary condition (3.3). A general solution of (3.8) is written which yields  $h(\rho, \zeta)$  for the region  $\rho \leq b$ . The three solutions possess six arbitrary constants. The constants are determined from the requirements that the field be finite at  $\rho = 0$ , that the field be regular at infinity and that it satisfy the radiation condition and the four boundary conditions (3.3), (3.4), (3.5), and (3.6). Choosing the solution  $h(\rho, \zeta)$  for region II, the inversion integral (2.7) must then be evaluated to yield  $H_\phi(\rho, z)$  for  $\rho \geq b$ .

Equation (3.7) is recognized as a form of Bessel's differential equation. It has the general solution

$$h(\rho, \zeta) = A J_1(\nu_1 \rho) + P Y_1(\nu_1 \rho)$$

where  $A$  and  $P$  are arbitrary constants,  $J_1(\nu_1 \rho)$  and  $Y_1(\nu_1 \rho)$  are Bessel functions of the first and second kind, respectively, and

$$\nu_1^2 = k_1^2 - \zeta^2.$$

Consider the case  $0 \leq \rho \leq a$ . Since  $H_\phi(\rho, z)$  and, therefore,  $h(\rho, \zeta)$  must be finite at  $\rho = 0$ , we determine that  $P = 0$ , since  $Y_1(\nu_1 \rho)$  is unbounded as  $\rho \rightarrow 0$ .

Hence

$$h(\rho, \zeta) = A J_1(\nu_1 \rho) \tag{3.9}$$

where

$$0 \leq \rho \leq a$$

$$\nu_1 = [k_1^2 - \zeta^2]^{1/2}.$$



For the region  $a \leq \rho \leq b$ , the  $Y_1$  function must be included, and we have

$$h(\rho, \zeta) = B J_1(\nu_1 \rho) + C Y_1(\nu_1 \rho) \quad (3.10)$$

where  $a \leq \rho \leq b$ ;

$$\nu_1 = [k_1^2 - \zeta^2]^{1/2}$$

For the region  $b \leq \rho$  we choose to write the general solution in terms of Hankel functions of the first and second kind

$$h(\rho, \zeta) = D H_1^{(1)}(\nu_0 \rho) + Q H_1^{(2)}(\nu_0 \rho)$$

where

$$\nu_0 = [k_0^2 - \zeta^2]^{1/2}$$

It will be seen later that, in general,  $\nu_0$  is complex; therefore we define the argument of  $\nu_0$  to be  $0 \leq \arg \nu_0 \leq \pi$ . As  $\rho \rightarrow \infty$ ,  $H_1^{(1)}(\nu_0 \rho)$  vanishes and  $H_1^{(2)}(\nu_0 \rho)$  is unbounded for the defined argument of  $\nu_0$ ; therefore  $Q = 0$  and

$$h(\rho, \zeta) = D H_1^{(1)}(\nu_0 \rho) \quad (3.11)$$

where

$$b \leq \rho$$

$$\nu_0 = [k_0^2 - \zeta^2]^{1/2}$$

$$0 \leq \arg \nu_0 \leq \pi.$$

In order to solve for the arbitrary constants A, B, C, and D, we apply the boundary conditions (3.3), (3.4), (3.5), and (3.6) to the



appropriate solutions (3.9), (3.10), and (3.11). Omitting the details, the following four equations are obtained:

$$\begin{aligned}
 & -(\nu_1 b) A J_1'(\nu_1 a) + (\nu_1 b) B J_1'(\nu_1 a) + (\nu_1 b) C Y_1'(\nu_1 a) = -i\omega \varepsilon_1 b \\
 & A J_1(\nu_1 a) - B J_1(\nu_1 a) - C Y_1(\nu_1 a) = 0 \\
 & B J_1(\nu_1 b) + C Y_1(\nu_1 b) - D H_1(\nu_0 b) = 0 \\
 & (\nu_1 b) B J_1'(\nu_1 b) + (\nu_1 b) C Y_1'(\nu_1 b) - D [\varepsilon_r(\nu_0 b) H_1'(\nu_0 b) + \\
 & \quad + (\varepsilon_r^{-1}) H_1(\nu_0 b)] = 0
 \end{aligned}
 \tag{3.12}$$

This system of equations was written in matrix form and solved by finding the inverse of the coefficient matrix. Details are given in Appendix A. The constant D was determined to be

$$D = i\omega \varepsilon_1 a \frac{J_1(\nu_1 a)}{(\nu_1 b) J_0(\nu_1 b) H_1^{(1)}(\nu_0 b) - \varepsilon_r(\nu_0 b) J_1(\nu_1 b) H_0^{(1)}(\nu_0 b)}
 \tag{3.13}$$

Substituting (3.13) into (3.11) we obtain  $h(\rho, \zeta)$  for the region  $\rho \geq b$ . The solution  $h(\rho, \zeta)$  may then be substituted into the inverse transform (2.7) to yield the integral expression for  $H_\phi(\rho, z)$  in region II. This result is

$$H_\phi(\rho, z) = \frac{i\omega \varepsilon_1 a}{2\pi} \int_{-\infty}^{+\infty} \frac{J_1(\nu_1 a) H_1^{(1)}(\nu_0 \rho) e^{+i\zeta z} d\zeta}{(\nu_1 b) J_0(\nu_1 b) H_1^{(1)}(\nu_0 b) - \varepsilon_r(\nu_0 b) J_1(\nu_1 b) H_0^{(1)}(\nu_0 b)}
 \tag{3.14}$$



where  $\rho \geq b$ ,

$$\nu_1 = [k_1^2 - \zeta^2]^{1/2}; \quad \nu_0 = [k_0^2 - \zeta^2]^{1/2}.$$

The evaluation of (3.14) is the subject of Chapter 5.



## 4. SINGULARITIES OF THE INTEGRAND--SOLUTION OF THE MODE EQUATION

The integral for  $H_\phi(\rho, z)$  is treated as a contour integral in the complex  $\zeta$  plane where the path of integration is along the real axis from  $-\infty$  to  $+\infty$ . In order to carry out the integration, it is first necessary to determine the singularities of the integrand. A detailed analysis of the singularities is presented in this chapter. The expression for  $H_\phi(\rho, z)$  is

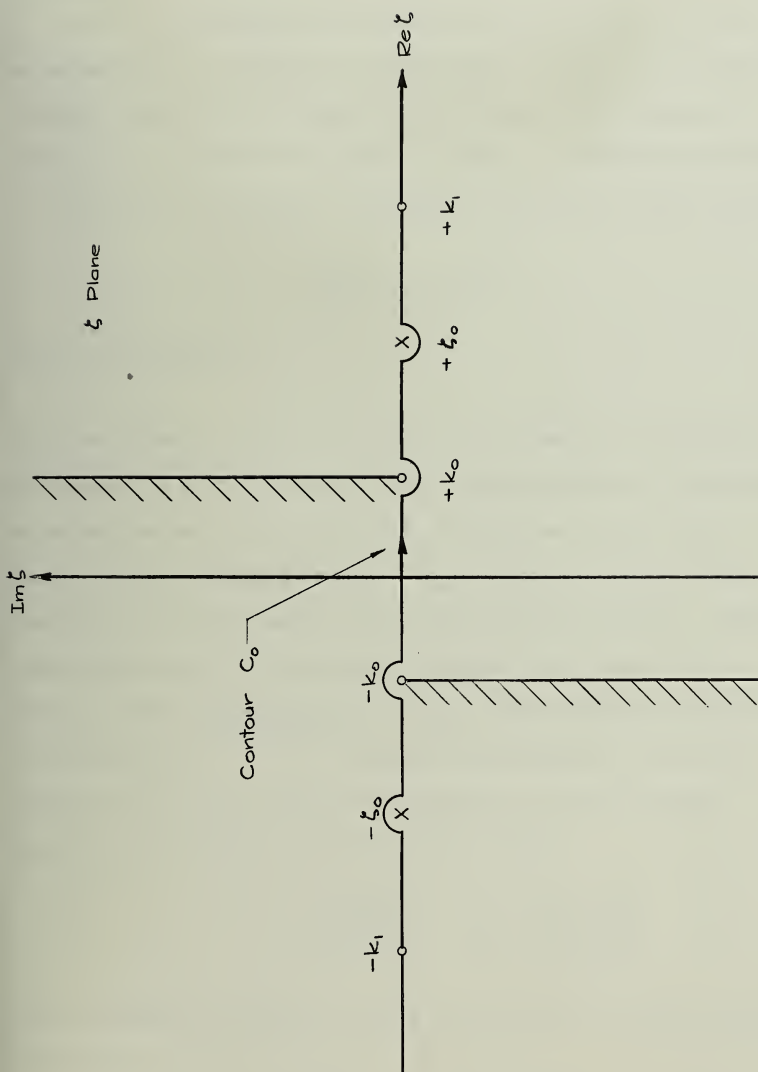
$$H_\phi(\rho, z) = \frac{+i\omega\epsilon_1 a}{2\pi} \int_{-\infty}^{+\infty} \frac{J_1(\nu_1 a) H_1^{(1)}(\nu_o \rho) e^{+i\zeta z} d\zeta}{(\nu_1 b) J_o(\nu_1 b) H_1^{(1)}(\nu_o b) - \epsilon_r(\nu_o b) J_1(\nu_1 b) H_o^{(1)}(\nu_o b)}$$

(4.1)

## 4.1 Branch Points

Consider the variable  $\nu_1$  which appears in the arguments of  $J_o$  and  $J_1$  in (4.1). Since  $\nu_1 = \sqrt{k_1^2 - \zeta^2}$ , then  $\nu_1$  is multiple-valued in any neighborhood of  $\zeta = \pm k_1$ . However, if one considers the power series expansions of  $J_o$  and  $J_1$ , one sees that the integrand is an even function of  $\nu_1$  so that the points  $\zeta = \pm k_1$  are not actually branch points. The Hankel function arguments contain the variable  $\nu_o = \sqrt{k_o^2 - \zeta^2}$ . The Hankel function has a logarithmic singularity at  $\nu_o = 0$ , and so is multiple-valued in any neighborhood of  $\zeta = \pm k_o$ . The points  $\zeta = \pm k_o$  are, therefore, branch points of the integrand. We select branch cuts in the  $\zeta$  plane as shown in Fig. 2. The points  $\pm k_1$  and  $\pm k_o$  lie on the real axis with  $|k_1| > |k_o|$  since  $\epsilon_1 > \epsilon_o$ . The branch cuts are defined as  $\zeta = \pm k_o + i \operatorname{Im} \zeta$ .



FIGURE 2. PATH OF INTEGRATION IN THE  $\zeta$  PLANE



## 4.2 Poles

Before we investigate the integrand of (4.1) for poles, it will be advantageous to consider the significance of a pole in the physical problem. Assume that the point  $\zeta = \zeta_0$  is a pole of the integrand. When (4.1) is evaluated as a contour integral, a residue contribution at  $\zeta = \zeta_0$  must be included. Thus,  $H_\phi(\rho, z)$  will have a term of the form

$$AH_1^{(1)}(\nu_0 \rho) e^{i\zeta_0 z} \quad (4.2)$$

where  $A = 2\pi i$  Residue  $\Big|_{\zeta = \zeta_0}$ .

Since we assumed a lossless dielectric rod by taking  $k^2 = \omega^2 \mu \epsilon$  real, we will expect (4.2) to represent a wave propagating without attenuation, which implies that  $\zeta_0$  is real. If  $\zeta_0$  is real and  $|\zeta_0| > |k_0|$ , then  $\nu_0$  is pure imaginary and  $H_1^{(1)}(\nu_0 \rho)$  reduces to the  $K_1$  Bessel function which decays exponentially with increasing argument. Under these conditions, (4.2) represents a surface wave of amplitude  $A$  having a radial dependence of  $H_1^{(1)}(\nu_0 \rho)$  which propagates along the rod according to  $e^{-i(\omega t - \zeta_0 z)}$ . We shall now determine if there are values of  $\zeta$  real which cause the denominator of (4.1) to vanish. Equating the denominator of (4.1) to zero yields

$$(\nu_1 b) \frac{J_0(\nu_1 b)}{J_1(\nu_1 b)} = \epsilon_r (\nu_0 b) \frac{H_0^{(1)}(\nu_0 b)}{H_1^{(1)}(\nu_0 b)} \quad (4.3)$$

For convenience of notation, let  $\nu_1 b = X_1$  and  $\nu_0 b = X_0$ ; then (4.3) may be written

$$X_1 \frac{J_0(X_1)}{J_1(X_1)} = \epsilon_r X_0 \frac{H_0^{(1)}(X_0)}{H_1^{(1)}(X_0)} \quad (4.4)$$



Recalling that

$$\nu_1 = \sqrt{k_1^2 - \zeta^2}$$

$$\nu_0 = \sqrt{k_0^2 - \zeta^2}$$

where  $k_1^2 > k_0^2 > 0$ ,

(i) Consider the case of  $\zeta$  real and  $|\zeta| > |k_1| > |k_0|$ . Then  $\nu_1$  and  $\nu_0$ , and therefore  $X_1$  and  $X_0$  will be pure imaginary. Denote  $X_1 = ip$  and  $X_0 = i\xi$ , where  $p$  and  $\xi$  are real; then (4.4) becomes

$$-(ip) \frac{J_0(ip)}{J_1(ip)} = \epsilon_r (-i\xi) \frac{H_0^{(1)}(i\xi)}{H_1^{(1)}(i\xi)}$$

which may be reduced to

$$-p \frac{I_0(p)}{I_1(p)} = \epsilon_r \xi \frac{K_0(\xi)}{K_1(\xi)}, \quad (4.5)$$

where  $I_n$  and  $K_n$  are modified Bessel functions of the first and second kinds, respectively. The left hand side of (4.5) is always negative for real  $p$  while the right side is always positive for real  $\xi$ , so there are no values  $(p, \xi)$  for which (4.5) can be satisfied. We conclude that no poles can occur on the real axis for  $|\zeta| > |k_1|$ .

(ii) Consider  $\zeta$  real and  $|\zeta| < |k_0| < |k_1|$ ; then  $\nu_1$  and  $\nu_0$ , and therefore  $X_1$  and  $X_0$  are pure real. From Eq. (4.4) we write

$$X_1 \frac{J_0(X_1)}{J_1(X_1)} = \epsilon_r X_0 \frac{H_0^{(1)}(X_0)}{H_1^{(1)}(X_0)} = -\epsilon_r X_0 \frac{H_0'(X_0)}{H_0^{(1)}(X_0)} \quad (4.6)$$

where  $H_0'$  denotes the derivative of  $H_0^{(1)}$ .



Using the definition of the Hankel functions, (4.6) may be written

$$X_1 \frac{J_0(X_1)}{J_1(X_1)} = -\epsilon_r X_0 \frac{J_0(X_0) + i Y_0(X_0)}{J_0'(X_0) + i Y_0'(X_0)} \quad (4.7)$$

Since  $X_1$  is real, the left side of (4.7) is always real, while the right side is obviously complex for real  $X_0$ . Is it possible for the right side to be real for some  $X_0$ ? The right side can be real only if the argument of the denominator is equal to that of the numerator by a multiple of  $n\pi$ .

Let

$$\alpha = \arg [J_0(X_0) + i Y_0(X_0)]$$

$$\beta = \arg [J_0'(X_0) + i Y_0'(X_0)]$$

Assume

$$\beta = \alpha + n\pi,$$

where

$$n = 0, \pm 1, \pm 2, \dots;$$

then

$$\tan \beta = \tan (\alpha + n\pi) = \frac{\tan \alpha + \tan n\pi}{1 - \tan \alpha \tan n\pi} = \tan \alpha$$

since

$$\tan \alpha = \frac{Y_0(X_0)}{J_0(X_0)} \quad \text{and} \quad \tan \beta = \frac{Y_0'(X_0)}{J_0'(X_0)}.$$

We obtain

$$\frac{Y_0(X_0)}{J_0(X_0)} = \frac{Y_0'(X_0)}{J_0'(X_0)}$$

or

$$J_0(X_0) Y_0'(X_0) - Y_0(X_0) J_0'(X_0) = 0. \quad (4.8)$$

However, the left side of (4.8) is the Wronskian of  $J_0$ ,  $Y_0$ , which is non-vanishing for finite  $X_0$ :

$$W \{J_0, Y_0\} = J_0(X_0) Y_0'(X_0) - Y_0(X_0) J_0'(X_0) = \frac{2}{\pi X_0} \neq 0.$$



We conclude that the right side of (4.7) must always be complex.

Since the left side of (4.7) can vanish, is it possible for the right side to be zero? For the right side to be zero,  $J_0(X_0)$  and  $Y_0(X_0)$  must be zero simultaneously. Watson<sup>9</sup> proves that the positive zeros of two distinct cylinder functions of the same order are interlaced; therefore,  $J_0(X_0)$  and  $Y_0(X_0)$  are never zero simultaneously. The same holds for  $-X_0$  since  $J_0(-X_0) = J_0(X_0)$  and  $Y_0(-X_0) = Y_0(X_0)$ . Since the right side of (4.7) is always complex and never vanishes, there are no real values  $X_1$  and  $X_0$  for which (4.6) can be satisfied. We conclude no poles exist for  $\xi$  real and  $|\xi| < |k_0|$ .

(iii) Consider  $\xi$  real and  $|k_0| < |\xi| < |k_1|$ ; then  $\nu_1$  is real,  $\nu_0$  is pure imaginary, and therefore  $X_1$  is real and  $X_0$  is pure imaginary. Let  $X_0 = i\xi$ , where  $\xi$  is positive and real. Equation (4.4) becomes

$$-X_1 \frac{J_0(X_1)}{J_1(X_1)} = \epsilon_r (-i\xi) \frac{H_0^{(1)}(i\xi)}{H_1^{(1)}(i\xi)} = \epsilon_r \xi \frac{K_0(\xi)}{K_1(\xi)}. \quad (4.9)$$

A graph of the functions  $-X_1 \frac{J_0(X_1)}{J_1(X_1)}$  and  $\epsilon_r \xi \frac{K_0(\xi)}{K_1(\xi)}$  is presented in Fig. 3.

The function  $-X_1 \frac{J_0(X_1)}{J_1(X_1)}$  is discontinuous each time  $J_1(X_1)$  vanishes

and so an infinity of branches occur for ever increasing  $X_1$ . It

passes through zero each time  $J_0(X_1)$  has a zero. The function

$\epsilon_r \xi \frac{K_0(\xi)}{K_1(\xi)}$  approaches zero for  $\xi \rightarrow 0$  and increases positively for

increasing  $\xi$ . It is evident in Fig. 3 that for every finite  $\xi$ , an

infinity of values  $X_1$  exist which satisfy Eq. (4.9). In order to

obtain a finite number of unique solutions of the mode equation,

a second relation between  $X_1$  and  $\xi$  is needed.



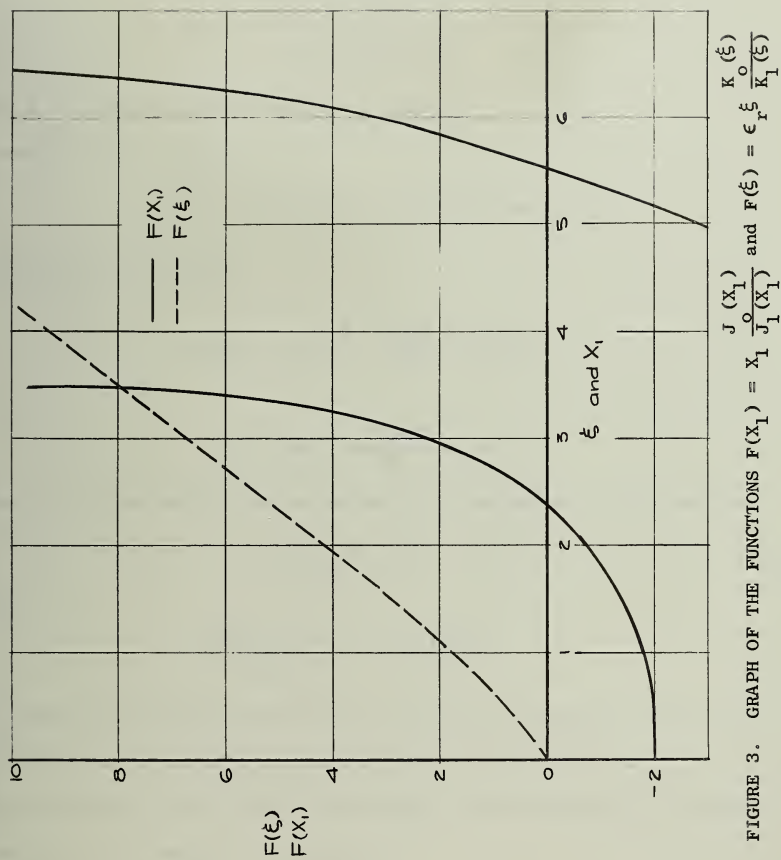


FIGURE 3. GRAPH OF THE FUNCTIONS  $F(X_1) = X_1 \frac{J_0(X_1)}{J_1(X_1)}$  and  $F(\xi) = \epsilon \xi \frac{K_0(\xi)}{K_1(\xi)}$



It follows from the definitions

$$\begin{aligned} \nu_1 &= \sqrt{k_1^2 - \zeta^2} \\ \nu_o &= \sqrt{k_o^2 - \zeta^2} \end{aligned}$$

where  $\zeta$  is identically the same in  $\nu_1$  and  $\nu_o$ .

Elimination of  $\zeta^2$  yields

$$\nu_1^2 - \nu_o^2 = k_1^2 - k_o^2 = \left(\frac{2\pi}{\lambda_o}\right)^2 (\epsilon_r - 1)$$

Multiplying by  $b^2$  yields

$$(\nu_1 b)^2 - (\nu_o b)^2 = \left(\frac{2\pi b}{\lambda_o}\right)^2 (\epsilon_r - 1)$$

or

$$X_1^2 - X_o^2 = \left(\frac{2\pi b}{\lambda_o}\right)^2 (\epsilon_r - 1)$$

Since  $X_o = i\xi$ , we obtain the second relation between  $X_1$  and  $\xi$ , which is

$$X_1^2 + \xi^2 = R^2 \quad (4.10)$$

where

$$R = \left(\frac{2\pi b}{\lambda_o}\right) \sqrt{\epsilon_r - 1} = k_o b \sqrt{\epsilon_r - 1}.$$

The graphical solution of the mode Eq. (4.9) is illustrated in Fig. 4, which includes curves of  $X_1$  as a function of  $\xi$  obtained from Fig. 3, and the relation (4.10). Note that (4.10) is the equation of a circle of radius  $R$  with center at the origin.

The multiple-valued solutions  $X_1$  for every  $\xi$  as discussed with Fig. 3 are evident in Fig. 4. The first branch of  $X_1$  starts at  $X_1 = 2.405$  for  $\xi = 0$  and approaches  $X_1 = 3.83$  asymptotically with increasing  $\xi$ .  $X_1 = 2.405$  is the first zero of  $J_o(X_1)$  and  $X_1 = 3.83$  is the first zero



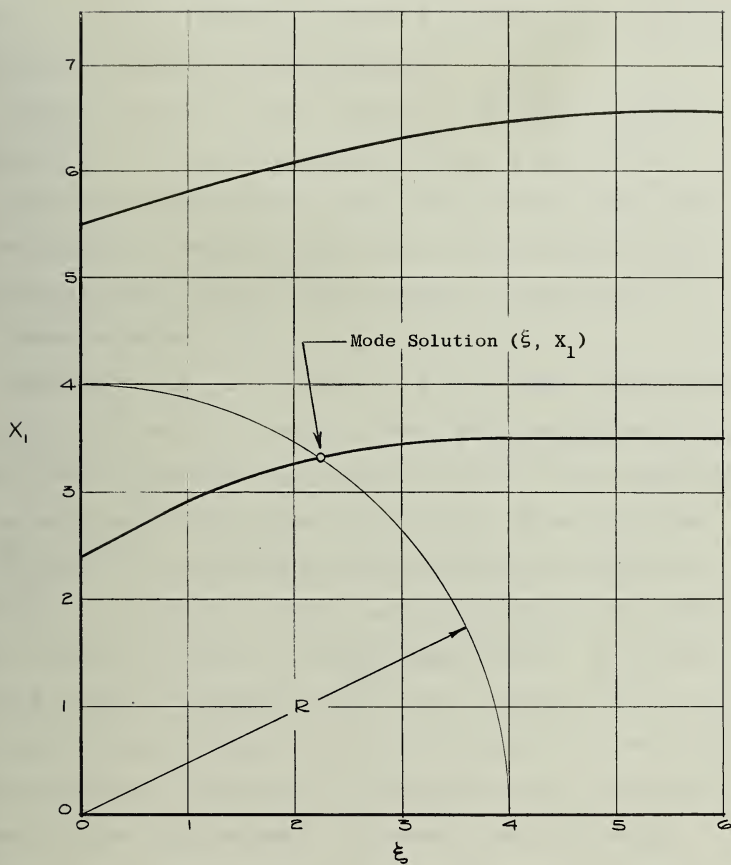


FIGURE 4. GRAPH OF  $X_1 = f(\xi)$  and  $X_1^2 + \xi^2 = R^2$



of  $J_1(X_1)$ . The second branch commences at 5.52 and approaches 7.02, which corresponds to the second zeros of  $J_0(X_1)$  and  $J_1(X_1)$ , respectively. An infinity of branches is thus established

Consider the case  $R < 2.405$ . Since  $X_1 = \sqrt{R^2 - \xi^2} \leq R$ , Fig. 3 shows that Eq. (4.9) cannot be satisfied for  $X_1 < 2.405$ , that is, no solution of the mode equation exists for  $R < 2.405$ . This is also evident in Fig. 4. Physically, this means that the dielectric rod waveguide is below cutoff and cannot propagate a surface wave of the transverse magnetic type.

Referring to Fig. 4, if  $2.405 < R < 5.52$ , a single, unique solution  $(\xi, X_1)$  of (4.9) and (4.10) results. Thus, (4.1) has poles at  $\zeta = \pm \zeta_0$ , where  $\zeta_0$  may be determined from the solution  $(\xi, X_1)$ . The poles occur on the real  $\zeta$  axis in the region  $|k_0| < |\zeta| < |k_1|$ . We see that when  $R = \left(\frac{2\pi b}{\lambda_0}\right) \sqrt{\epsilon_r - 1}$  is restricted to the range  $2.405 < R < 5.52$ , the dielectric rod propagates a single, surface wave which is the lowest order, circularly symmetric, transverse magnetic mode. It is known as the  $E_{01}$  mode and propagates as  $e^{i\zeta_0 z}$  for  $z > 0$  and  $e^{-i\zeta_0 z}$  for  $z < 0$ .

As  $R$  increases, the number of solutions  $(\xi, X_1)$  of (4.9) and (4.10) also increases, and the dielectric rod supports an ever increasing number of higher order TM modes. The number of poles occurring between  $k_0$  and  $k_1$  increases with  $R$ , as is shown in Fig. 5. It is interesting to note the behavior of  $k_0$ ,  $k_1$ , and the poles as  $R$  increases.

From the definition of  $\nu_0$  we obtain

$$\zeta = \sqrt{k_0^2 + \left(\frac{\xi}{b}\right)^2} \quad (4.11)$$



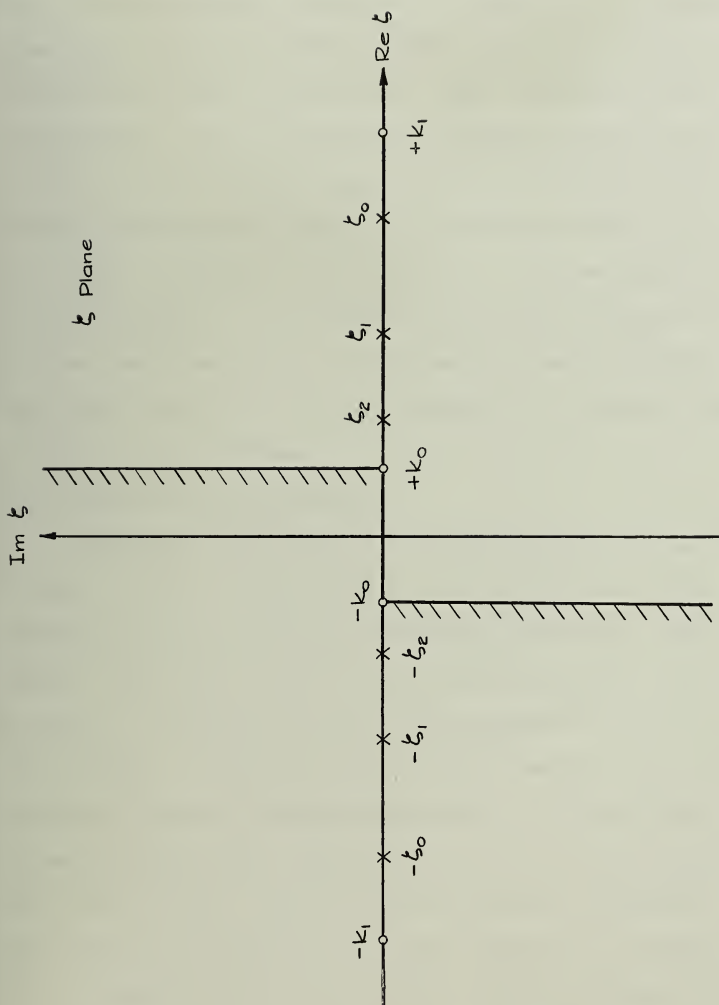


FIGURE 5. DISTRIBUTION OF POLES ON THE REAL AXIS



Consider  $R$  just greater than 2.405; then  $X_1 \approx 2.405$  and  $\xi$  is a very small positive quantity. Equation (4.11) shows  $|\xi|$  is just greater than  $|k_0|$ , that is, the poles  $\pm \xi$  first appear in the neighborhood of the branch points  $\pm k_0$  with  $|\xi| > |k_0|$ . Referring to Fig. 4, as  $R$  increases, the  $\xi$  of any solution increases, so that  $|\xi|$  corresponding to  $(\xi, X_1)$  also increases. Thus, the poles  $\pm \xi$  move away from the origin. Each new pair of poles  $\pm \xi_1$  first appear near  $\xi = \pm k_0$ . Now  $k_0 = (\frac{2\pi}{\lambda_0})$ ,  $k_1 = \sqrt{\epsilon_r} k_0$ , and  $R = k_0 b \sqrt{\epsilon_r - 1}$ , so we may think of  $k_1$  and  $k_0$  as proportional to  $R$ . The distance from  $k_1$  to  $k_0$ ,  $(k_1 - k_0) = k_0 (\sqrt{\epsilon_r} - 1)$  is also proportional to  $R$ . As  $R$  increases,  $k_1$  and  $k_0$  move away from the origin, the interval  $(k_1 - k_0)$  increases, and the number of poles between  $k_1$  and  $k_0$  increases.

#### 4.3 Solution of the Mode Equation

The object of this work is to determine the efficiency of a magnetic current filament in exciting the  $E_{01}$  mode on a dielectric rod. For single mode operation,  $R$  must be restricted to the range  $2.405 < R < 5.52$ . Since  $R = k_0 b \sqrt{\epsilon_r - 1}$ , this restriction on  $R$  is actually a limitation on  $k_0 b$  and  $\epsilon_r$ , the physical constants which describe the dielectric rod as a waveguide. Specific solutions of (4.9) and (4.10) are required in order to calculate the power transmitted in the surface wave and, ultimately, efficiency. Equations (4.9) and (4.10) were solved for the case  $\epsilon_r = 2.56$ , a relative dielectric constant typical for polystyrene. Six solutions  $(\xi, X_1)$  were obtained for selected values of  $k_0 b$  equal to 2.2, 2.6, 3.0, 3.4, 3.8, and 4.2. Graphical solution of (4.9) and (4.10) as illustrated in Fig. 4 was considered too



inaccurate for the purposes of numerical computation and so (4.9) and (4.10) were solved, utilizing the University of Illinois digital computer. The solutions ( $\xi$ ,  $X_1$ ) are accurate to four decimal places. Details of the method used to solve the equations on the computer are given in Appendix B. Table I presents  $\xi$ ,  $X_1$ , and the ratio  $\lambda_g/\lambda_0$  corresponding to each value of  $k_0 b$ . The ratio  $\lambda_g/\lambda_0$  follows from the definition of  $\nu_0$ .

$$\lambda_g/\lambda_0 = \frac{1}{\sqrt{1 + (\frac{\xi}{k_0 b})^2}} \quad (4.12)$$

TABLE I

| $k_0 b$ | $\xi$  | $X_1$  | $\lambda_g/\lambda_0$ |
|---------|--------|--------|-----------------------|
| 2.2     | 5603   | 2.6901 | .9691                 |
| 2.6     | 1.2329 | 3.0043 | .9036                 |
| 3.0     | 1.9353 | 3.2086 | .8403                 |
| 3.4     | 2.6269 | 3.3366 | .7913                 |
| 3.8     | 3.2905 | 3.4204 | .7560                 |
| 4.2     | 3.9264 | 3.4788 | .7305                 |



## 5. EVALUATION OF THE CONTOUR INTEGRAL

For the purposes of the following discussion, we repeat the integral solution for  $H_\phi(\rho, z)$  outside the dielectric rod, which is

$$H_\phi(\rho, z) = \frac{+i\omega\epsilon_1 a}{2\pi} \int_{C_0} \frac{J_1(\nu_1 a) H_1^{(1)}(\nu_0 \rho) e^{+i\zeta z}}{J_0(\nu_1 b) J_0(\nu_1 b) H_1^{(1)}(\nu_0 b) - \epsilon_r(\nu_0 b) J_1(\nu_1 b) H_0^{(1)}(\nu_0 b)} d\zeta \quad (5.1)$$

where  $C_0$  is the contour along the real axis with appropriate indentations at the branch points  $\pm k_0$  and poles  $\pm \zeta_0$ , as shown in Fig. 2. Expression (5.1) is evaluated by transforming from the  $\zeta$  plane to the complex  $\tau$  plane and employing the saddle point method of integration to obtain an asymptotic value of the integral.<sup>10,11</sup> The saddle point method is applicable only when certain parameters in the integrand are relatively large. It is used to obtain an approximation to the far zone radiation field. The surface wave field follows from the residue contribution at the pole. In the  $\tau$  plane the contour of integration is deformed until it passes through the saddle point of the integrand. The integrand contains an exponential function, and when the deformed contour corresponds to the path of "steepest descent" the exponential function decreases very rapidly away from the saddle point. The integrand vanishes at the ends of the path, and so the principal contribution to the integral comes from evaluation along the path of steepest descent in the very near vicinity of the saddle point.

### 5.1 The Radiation Field

In order to apply the saddle point method of integration, we introduce



the transformation of variable

$$\zeta = k_0 \sin \tau$$

where

$$\tau = \psi + i\eta \quad (5.2)$$

Now  $\tau = \sin^{-1}(\zeta/k_0)$  is a multiple-valued function of  $\zeta$  and the region of integration in the  $\zeta$  plane transforms into a strip in the  $\tau$  plane bound by two curved lines which correspond to the branch cuts in the  $\zeta$  plane.

They are defined by the equations

$$\begin{cases} \sin \psi \cosh \eta = 1 \\ \cos \psi \sinh \eta = \frac{\text{Im } \zeta}{k_0} \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} \sin \psi \cosh \eta = -1 \\ \cos \psi \sinh \eta = \frac{\text{Im } \zeta}{k_0} \leq 0 \end{cases} \quad (5.3)$$

The contour of integration  $C_0$  in the  $\zeta$  plane, which is defined as  $\text{Im } \zeta = 0$  with necessary detours, transforms to the path  $C_1$  in the  $\tau$  plane, as shown in Fig. 6. It follows from the definition  $\text{Im } \zeta = 0$  which corresponds to  $k_0 \cos \psi \sinh \eta = 0$  in the  $\tau$  plane while maintaining  $C_1$  within the curved strip. The direction of  $C_1$  follows from consideration of the real part of  $\zeta$  along  $C_0$ , where  $\text{Re } \zeta = k_0 \sin \psi \cosh \eta$ . The branch points  $\zeta = \pm k_0$  transform to the points  $\tau = \pm \pi/2$  while the images of the poles  $\pm \zeta_0$  are the points  $\tau_0 = +\pi/2 - i \cosh^{-1} \zeta_0/k_0$  and  $-\tau_0 = -\pi/2 + i \cosh^{-1} \zeta_0/k_0$ . Substituting (5.2) into the expression for  $\nu_0$  yields  $\nu_0 = \sqrt{k_0^2 - \zeta^2} = \pm k_0 \cos \tau$ . The choice of sign here is determined by the previous definition of  $\arg \nu_0$  in the  $\zeta$  plane. On the contour  $C_0$ ,  $\zeta$  is real and when  $|\zeta| > |k_0|$  we obtain  $\nu_0 = |\nu_0| e^{\pm i \pi/2}$ . Since we chose to define  $0 \leq \arg \nu_0 \leq \pi$ , the negative sign is excluded and we have  $\nu_0 = |\nu_0| e^{+i \pi/2}$ . In order that corresponding values of  $\tau$  on  $C_1$  yield



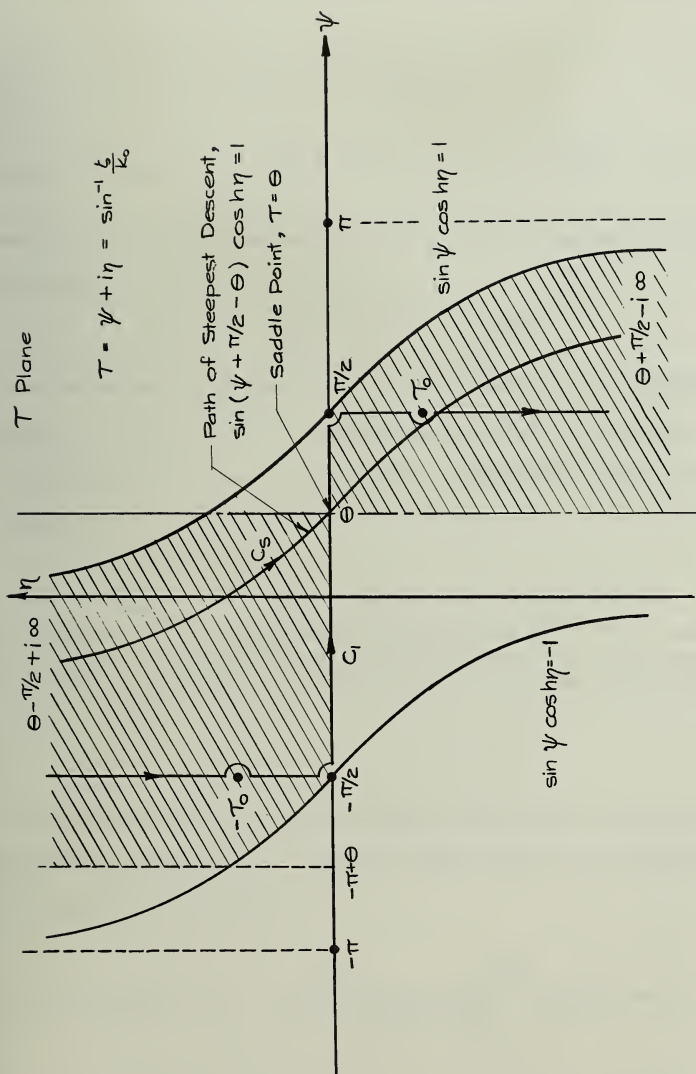


FIGURE 6. PATH OF INTEGRATION IN THE  $\tau$  PLANE



the same argument for  $\nu_0$ , the positive sign is required; thus we establish

$$\nu_0 = + k_0 \cos \tau. \quad (5.4)$$

The function  $\nu_1$  becomes  $\nu_1 = \sqrt{k_1^2 - \zeta^2} = \pm k_0 \sqrt{\epsilon_r - \sin^2 \tau}$ . Since the

integrand of (5.1) is an even function of  $\nu_1$ , either sign may be used and we choose

$$\nu_1 = + k_0 \sqrt{\epsilon_r - \sin^2 \tau} = k_0 w(\tau) \quad (5.5)$$

where for convenience of notation we let  $w$ , a function of  $\tau$ , represent

the radical,  $w = \sqrt{\epsilon_r - \sin^2 \tau}$ . Substitution of (5.2), (5.4), and (5.5)

into (5.1) with  $d\zeta = k_0 \cos \tau d\tau$  yields the transformed integral in the  $\tau$  plane which is

$$H_\phi(\rho, z) = \frac{i\omega\epsilon_1 a}{2\pi b} \int_{C_1} \frac{J_1(k_0 a w) H_1^{(1)}(k_0 \rho \cos \tau) e^{ik_0 z \sin \tau} \cos \tau d\tau}{w J_0(k_0 b w) H_1^{(1)}(k_0 b \cos \tau) - \epsilon_r \cos \tau J_1(k_0 b w) H_0^{(1)}(k_0 b \cos \tau)} \quad (5.6)$$

If  $\rho$  is large and  $k_0 \rho \cos \tau \neq 0$ , the Hankel function may be replaced with its asymptotic representation. Then the Hankel function and exponential in (5.6) combine in the form

$$H_1^{(1)}(k_0 \rho \cos \tau) e^{ik_0 z \sin \tau} \sim e^{-i\frac{3\pi}{4}} \left\{ \frac{2}{\pi k_0 \rho \cos \tau} \right\}^{1/2} e^{ik_0(\rho \cos \tau + z \sin \tau)} \quad (5.7)$$



Consider the spherical coordinate system illustrated in Fig. 7, where the

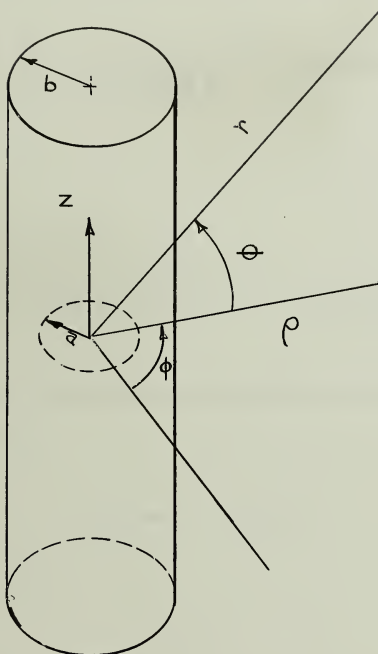


FIGURE 7 CIRCULAR CYLINDRICAL COORDINATE SYSTEM  $(\rho, \phi, z)$  AND SPHERICAL COORDINATE SYSTEM  $(r, \phi, \theta)$

polar angle  $\theta$  is measured from the plane  $z = 0$ . In this coordinate system we note that  $\rho = r \cos \theta$  and  $z = r \sin \theta$ . Substituting for  $\rho$  and  $z$  in the right side of (5.7) we obtain

$$H_1^{(1)}(k_0 \rho \cos \gamma) e^{ik_0 z \sin \gamma} \sim e^{-i\frac{3\pi}{4}} \left[ \frac{2}{\pi k_0 r \cos \theta \cos \gamma} \right]^{1/2} e^{ik_0 r \cos(\gamma - \theta)}. \quad (5.8)$$

We substitute (5.8) into (5.6) to obtain the following approximation for



for  $H_\phi(r, \theta)$  which holds for  $r$  large and  $k_o r \cos \theta \neq 0$ .

$$H_\phi(r, \theta) = \frac{i\omega\epsilon_a a}{2\pi b} e^{-i\frac{3\pi}{4}} \int_{C_1} F(\gamma) \left[ \frac{2 \cos \tau}{\pi k_o r \cos \theta} \right]^{\frac{1}{2}} e^{i k_o r \cos(\tau - \theta)} d\tau \quad (5.9)$$

where

$$F(\tau) = \frac{J_1(k_o a w)}{w J_0(k_o b w) H_1^{(1)}(k_o b \cos \tau) - \xi_r \cos \tau J_1(k_o b w) H_0^{(1)}(k_o b \cos \tau)}$$

Integral (5.9) is readily evaluated by the saddle point method. The saddle point is found by setting

$$\frac{d}{d\tau} \cos(\tau - \theta) = 0 \quad (5.10)$$

which gives

$$\tau = \theta. \quad (5.11)$$

Note that the saddle point corresponds to the polar angle  $\theta$ . The path of steepest descent is defined by

$$\text{Im} [i \cos(\tau - \theta)] = \cos(\psi - \theta) \cosh \eta = 1,$$

which may be written

$$\sin(\psi + \pi/2 - \theta) \cosh \eta = 1. \quad (5.12)$$

Thus, the steepest descent path  $C_s$  has the same form as the curve bounding the right side of the strip, but it is displaced to the left by the amount  $(\pi/2 - \theta)$ , as shown in Fig. 6. One may verify that on  $C_s$ ,  $\frac{d\eta}{d\psi} = -1$  at  $\gamma = \theta$ ,



so that the steepest descent path crosses the real axis at the angle  $-\pi/4$ . We deform contour  $C_1$  into the path  $C_s$  which starts at  $\theta - \pi/2 + i\infty$  and ends at  $\theta + \pi/2 - i\infty$ . In order for (5.9) to converge as  $\eta$  approaches  $\pm\infty$  on  $C_s$ , it is necessary that

$$\operatorname{Re} [i \cos (\gamma - \theta)] = \sin (\psi - \theta) \sinh \eta < 0. \quad (5.13)$$

This condition is satisfied in the region

$$\left. \begin{array}{l} \eta > 0 \text{ for } -\pi + \theta < \psi < \theta \\ \eta < 0 \text{ for } \theta < \psi < \pi \end{array} \right\} \quad (5.14)$$

and

The shaded area of Fig. 6 is the region of convergence and we see that the integrand of (5.9) does vanish on  $C_s$  as  $\eta$  approaches  $\pm\infty$ .

The relation of the path of steepest descent and the pole  $\gamma_0$  will now be considered. Evaluation of (5.9) along  $C_s$  yields the total field in the region outside the dielectric rod. Since the saddle point is equal to the polar angle  $\theta$ , as  $\theta$  increases towards  $\pi/2$ , the steepest descent path shifts to the right in Fig. 6. Ultimately, a residue term must be included because  $C_s$  will cross to the right of the pole  $\gamma_0$ . Therefore, the total field outside the rod, which is obtained as  $\theta$  takes on values  $0 \leq \theta < \pi/2$ , may be considered as the sum of the residue at the pole and the steepest descent contribution at the saddle point, that is, the surface wave and the radiation field.

Due to the rapid decay of the exponential term away from the saddle point, it is only necessary to integrate over a short segment of  $C_s$  in the near vicinity of  $\gamma = \theta$ .



We proceed by setting

$$\begin{aligned}\gamma - \theta &= \sigma e^{-i\frac{\pi}{4}} \\ d\gamma &= d\sigma e^{-i\frac{\pi}{4}}\end{aligned}\quad (5.15)$$

where  $\sigma$  is the distance along  $C_s$  measured from the saddle point. Considering  $|\gamma - \theta|$  small, we may approximate

$$\cos(\gamma - \theta) \approx 1 - \frac{(\gamma - \theta)^2}{2} = 1 - \frac{\sigma^2}{2} e^{-i\frac{\pi}{2}} \quad (5.16)$$

Except for the exponential term, the integrand of (5.9) is slowly varying about  $\gamma = \theta$  and may be assumed as constant. We substitute (5.15) and (5.16) into (5.9) to obtain

$$H_\phi(r, \theta) = \frac{-i\omega \epsilon_1^a}{2\pi b} \left[ \frac{2}{\pi k_o r} \right]^{1/2} F(\theta) e^{ik_o r} \int_{-\Delta}^{+\Delta} e^{-\frac{k_o r \sigma^2}{2}} d\sigma. \quad (5.17)$$

Since  $r$  is very large, the following approximation is valid

$$\int_{-\Delta}^{+\Delta} e^{-\frac{k_o r \sigma^2}{2}} d\sigma \approx \int_{-\infty}^{+\infty} e^{-\frac{k_o r \sigma^2}{2}} d\sigma = \left[ \frac{2\pi}{k_o r} \right]^{1/2} \quad (5.18)$$

Substituting (5.18) into (5.17) and noting that  $\omega \epsilon_1 = k_o \epsilon_r \sqrt{\frac{\epsilon_o}{\mu_o}}$ , we obtain the far zone radiation field, which is

$$H_\phi(r, \theta) = -i \sqrt{\frac{\epsilon_o}{\mu_o}} \left( \frac{\epsilon_r^a}{\pi b} \right) F(\theta) \frac{e^{ik_o r}}{r} \quad (5.19)$$

where

$$F(\theta) = \frac{J_1(k_o a w)}{w J_0(k_o b w) H_1^{(1)}(k_o b \cos \theta) - \epsilon_r \cos \theta J_1(k_o b w) H_0^{(1)}(k_o b \cos \theta)}$$



and

$$w = \sqrt{\xi_r - \sin^2 \theta}$$

It follows from Maxwell's equations that

$$E_{\theta}(r, \theta) = -i \left( \frac{\xi_r^a}{\pi b} \right) F(\theta) \frac{e^{ik_0 r}}{r} \quad (5.20)$$

## 5.2 The Surface Wave Field

The surface wave propagating in the positive  $z$  direction is obtained by evaluating the residue of (5.1) at the pole  $\zeta_0$ . Denote the denominator of (5.1) by  $G(\zeta)$  so that (5.1) may be written

$$H_{\phi}(\rho, z) = \int_{C_0} \frac{-i\omega \xi_1 a}{2\pi} \frac{J_1(\nu_1 a) H_1^{(1)}(\nu_0 \rho) e^{i\zeta z}}{G(\zeta)} d\zeta \quad (5.21)$$

where

$$G(\zeta) = (\nu_1 b) J_0(\nu_1 b) H_1^{(1)}(\nu_0 b) - \xi_r(\nu_0 b) J_1(\nu_1 b) H_0^{(1)}(\nu_0 b)$$

The integral (5.21) is the solution for the total field  $H_{\phi}(\rho, z)$  in Region II where  $\rho \geq b$ . Let  $H_{\phi}^{II}(\rho, z)$  represent the surface wave portion of the field in Region II. It is given by

$$H_{\phi}^{II}(\rho, z) = 2\pi i \text{ Residue} \left[ \frac{-i\omega \xi_1 a}{2\pi} \frac{J_1(\nu_1 a) H_1^{(1)}(\nu_0 \rho) e^{i\zeta z}}{G(\zeta)} \right]_{\zeta = \zeta_0} \quad (5.22)$$

where  $z > 0$ .

The pole  $\zeta_0$  follows from the solution  $(\xi, X_1)$  of the mode equation  $G(\zeta) = 0$ .

Since  $G'(\zeta_0) = \frac{d}{d\zeta} G(\zeta) \Big|_{\zeta = \zeta_0} \neq 0$ ,  $\zeta_0$  is a simple pole and the residue of



of the integrand at  $\zeta_0$  is given by

$$\text{Res} \Big|_{\zeta_0} = \frac{i\omega \varepsilon_1 a}{2\pi} \cdot \frac{J_1\left(\frac{a}{b} X_1\right) H_1^{(1)}\left(i \xi \frac{\rho}{b}\right) e^{i\zeta_0 z}}{G'(\zeta_0)} \quad (5.23)$$

Using Bessel function recurrence relations,  $G'(\zeta_0)$  was determined as

$$G'(\zeta_0) = \frac{2}{\pi} \zeta_0 b^2 \left\{ 2 \frac{(R/\xi)^2}{X_1} J_0(X_1) K_1(\xi) + \left[ \frac{X_1}{\xi} + \varepsilon_r \left( \frac{\xi}{X_1} \right) \right] J_0(X_1) K_0(\xi) + \right. \\ \left. + (\varepsilon_r - 1) J_1(X_1) K_1(\xi) \right\} \quad (5.24)$$

Using the relation  $H_1^{(1)}\left(i \xi \frac{\rho}{b}\right) = -\frac{2}{\pi} K_1\left(\xi \frac{\rho}{b}\right)$ , it follows from (5.22), (5.23), and (5.24) that

$$H_\phi^{II}(\rho, z) = A_{II} K_1\left(\xi \frac{\rho}{b}\right) e^{i\zeta_0 z} \quad (5.25)$$

where  $\rho \geq b$ ,  $z > 0$

$$A_{II} = \frac{\omega \varepsilon_1 \left(\frac{a}{b}\right) J_1\left(\frac{a}{b} X_1\right)}{\zeta_0 b^2 \left\{ 2 \frac{(R/\xi)^2}{X_1} J_0(X_1) K_1(\xi) + \left[ \varepsilon_r \left( \frac{\xi}{X_1} \right) + \frac{X_1}{\xi} \right] J_0(X_1) K_0(\xi) + (\varepsilon_r - 1) J_1(X_1) K_1(\xi) \right\}}$$

with

$$\zeta_0 b = \sqrt{(k_0 b)^2 + \xi^2}$$

and

$$R^2 = X_1^2 + \xi^2.$$

It is interesting to note that the surface wave solution (5.25) satisfies Maxwell's equations but its Fourier transform (2.6) does not exist since  $\zeta_0$  is real. This situation results from the simplifying



assumption that the dielectric rod is lossless. If the finite loss of the rod is included, the constant  $k_1^2 = \omega^2 \mu_0 \epsilon_1 (1 + i \frac{\sigma}{\omega \epsilon_1})$  is complex since  $\sigma \neq 0$ . It follows that the poles  $\pm \zeta_0$ , which are defined by the solution of the mode equation for complex arguments, no longer lie on the real axis. Instead, they appear in the first and third quadrants of the  $\zeta$  plane and the Fourier transforms of the solutions corresponding to the complex poles  $\pm \zeta_0$  exist. In the limit as the loss approaches zero ( $\sigma \rightarrow 0$ ), the poles lie on the real axis. The surface wave solutions still satisfy the homogeneous, scalar wave equation but their Fourier transform no longer converges. We have treated the limiting case where  $\sigma$  vanishes in order to be able to solve the mode equation. This approximation is justified since, in practice, the conductivity  $\sigma$  of the dielectric is extremely small.

We know from the solution of Maxwell's equations for the source free dielectric rod that the surface wave in Region I is

$$H_{\phi}^I(\rho, z) = A_I J_1(\nu_1 \rho) e^{i\zeta_0 z} = A_I J_1(x_1 \frac{\rho}{b}) e^{i\zeta_0 z} \quad (5.26)$$

Tangential  $\vec{H}$  must be continuous at  $\rho = b$  for all  $z$ , so

$$H_{\phi}^I(b, z) = H_{\phi}^{II}(b, z).$$

Equating (5.25) and (5.26) at  $\rho = b$  yields

$$A_I = \frac{K_1(\xi)}{J_1(x_1)} A_{II}.$$

Therefore

$$H_{\phi}^I(\rho, z) = A_{II} \frac{K_1(\xi)}{J_1(x_1)} J_1(x_1 \frac{\rho}{b}) e^{i\zeta_0 z} \quad (5.27)$$



We obtain  $E_\rho(\rho, z)$  in Regions I and II from  $\vec{E} = \frac{1}{-i\omega\epsilon} \nabla \times \vec{H}$ , hence

$$E_\rho^I(\rho, z) = \left( \frac{1}{i\omega\epsilon_1} \right) \frac{\partial H_\phi^I}{\partial z} = \left( \frac{\zeta_0}{\omega\epsilon_1} \right) H_\phi^I(\rho, z) \quad (5.28)$$

$$E_\rho^{II}(\rho, z) = \left( \frac{\zeta_0}{\omega\epsilon_0} \right) H_\phi^{II}(\rho, z) \quad (5.29)$$

The field components (5.25), (5.27), and (5.28), and (5.29) will be used to calculate the surface wave power in Chapter 6.



## 6. THE POWER INTEGRALS AND EXCITATION EFFICIENCY

In the previous section, the surface wave field and radiation field generated by the magnetic current source were determined. It is now possible to calculate the corresponding powers and obtain the efficiency of the source. The efficiency with which the source delivers power to the surface wave is called the excitation efficiency of the source. Denoting efficiency by the symbol  $\Upsilon$ , it is defined as

$$\Upsilon = \frac{W^S}{W^T} \quad (6.1)$$

where  $W^S$  is the surface wave power

and  $W^T$  is the total power delivered by the source.

Goubau<sup>12</sup> has proved that, for lossless surface waveguides, the radiation and surface wave fields are orthogonal with regard to power considerations. In this case, the total energy delivered by the source is equal to the sum of the surface wave power and the radiated power. Since we are considering a lossless dielectric rod, the orthogonality condition holds and  $\Upsilon$  is given by

$$\Upsilon = \frac{W^S}{W^S + W^R} \quad (6.2)$$

where  $W^S$  is the surface wave power

and  $W^R$  is the radiated power.

In order to determine  $\Upsilon$ , we proceed with the calculation of  $W^R$  and  $W^S$ .



## 6.1 Radiated Power

The radiated power  $W^R$  is obtained by integrating the time average Poynting vector over the surface of a large sphere of radius  $r$ .

$$W^R = \int \int \bar{\mathbf{P}}_{av} \cdot \bar{d\mathbf{a}} = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1}{2} \operatorname{Re} \bar{\mathbf{E}} \times \bar{\mathbf{H}}^* \cdot \bar{d\mathbf{a}} \quad (6.3)$$

where  $d\mathbf{a} = r^2 \cos \theta \, d\theta \, d\phi$ .

The radiation field, which has components  $E_\theta$  and  $H_\phi$ , is independent of  $\phi$  and is symmetrical about the plane  $\theta = 0$ , so (6.3) reduces to

$$W^R = 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} \operatorname{Re} E_\theta H_\phi^* (2\pi r^2 \cos \theta) \, d\theta. \quad (6.4)$$

Substitution of (5.19) and (5.20) into (6.4) yields the integral for the radiated power

$$W^R = \sqrt{\frac{\epsilon_o}{\mu_o}} \left( \frac{\epsilon_r a}{b} \right)^2 \frac{1}{(\pi/2)} \int_0^{\frac{\pi}{2}} \cos \theta \, |F(\theta)|^2 \, d\theta. \quad (6.5)$$

where

$$F(\theta) = \frac{J_1(k_o a w)}{w J_o(k_o b w) H_1^{(1)}(k_o b \cos \theta) - \epsilon_r \cos \theta J_1(k_o b w) H_o^{(1)}(k_o b \cos \theta)}$$

$$w = \sqrt{\epsilon_r - \sin^2 \theta}.$$

A discussion of the integrand of (6.5) is worthwhile at this point.

We note that the magnitude of the radiation field is proportional to

$|F(\theta)|$ . As  $\theta$  approaches  $\pi/2$ ,  $|F(\theta)|$  vanishes so the radiation pattern



has a null along the axis of the dielectric rod. The influence of the source dimension ( $k_o a$ ) on the radiation field and  $W^R$  is contained in the function  $J_1(k_o a w)$  which is the numerator of  $F(\theta)$ . The dielectric rod parameter ( $k_o b$ ) appears only in the denominator. In order to obtain maximum excitation efficiency for any  $k_o b$ ,  $W^R$  should be a minimum.  $W^R$  will be a minimum if  $J_1(k_o a w)$  vanishes or is quite small as  $\theta$  ranges from zero to  $\pi/2$ ; this occurs when  $J_1(k_o a w)$  passes through a zero. Consider the function  $w = \sqrt{\epsilon_r - \sin^2 \theta}$  with  $\epsilon_r = 2.56$ . As  $\theta$  takes on values  $0 \leq \theta \leq \pi/2$ ,  $w$  has the range  $1.25 \leq w \leq 1.6$ . Assume an average value for  $w$ ,  $w_{av} = 1.42$ . We should expect the most efficient source to have the dimension  $k_o a \approx 3.83/w_{av} = 2.7$ , where 3.83 is the first zero of the  $J_1$  function. This analysis is verified in the efficiency data presented in the latter part of this section.

The integral portion of (6.5) was evaluated numerically on the University of Illinois digital computer. Since the denominator of  $F(\theta)$  is unbounded as  $\theta$  approaches  $\pi/2$ , the upper limit of integration was selected as  $88^\circ$  and the range of integration was subdivided into 44 equal intervals of  $2^\circ$  each. The computer integration routine evaluated the integral by Simpson's rule. The computer program is discussed more fully in Appendix C. The constant  $\sqrt{\frac{\epsilon_o}{\mu_o} \left( \frac{\epsilon_r a^2}{b} \right)}$  which multiplies the integral in (6.5) also occurs in  $W^S$  and may be cancelled in the evaluation of  $\Upsilon$ . The quantity that was determined on the computer and which was used to calculate  $\Upsilon$  is the normalized integral

$$N^R = \frac{1}{\pi/2} \int_0^{\frac{88}{90} \frac{\pi}{2}} \cos \theta |F(\theta)|^2 d\theta \quad (6.6)$$



where

$$W^R = \sqrt{\frac{\epsilon_o}{\mu_o}} \left( \frac{\epsilon_r a}{b} \right)^2 N^R.$$

$N^R$  was computed as  $k_o a$  varied in discrete steps of 0.2 over the range  $0 < k_o a \leq k_o b$  for the selected value of  $k_o b$  under consideration. The results are given in Table II, along with  $N^S$  and the excitation efficiency  $\Upsilon$ .

## 6.2 Surface Wave Power

The total power in the surface wave launched by the magnetic current source is determined in the following manner. Consider an infinite transverse plane normal to the  $z$  axis located at some position  $z$  positive. The power propagating in the positive  $z$  direction is obtained by integrating the time average Poynting vector over the surface of the infinite plane. Because of symmetry, the power propagating in the negative  $z$  direction is equal to the power propagating in the positive  $z$  direction. Therefore, the total surface wave power is equal to twice the power in the positive  $z$  direction. Since the field components  $E_\rho$ ,  $H_\phi$  are different for Regions I and II, separate integrations are necessary for the regions  $0 \leq \rho \leq b$  and  $\rho \geq b$ . Consider the positive  $z$  direction. We denote the power over the cross sectional area of the rod as  $W^I$  and  $W^{II}$  applies to region II, which extends to infinity. The total surface wave power  $W^S$  is given by

$$W^S = 2 (W^I + W^{II}). \quad (6.7)$$

Consider region I where  $0 \leq \rho \leq b$ , the integral for  $W^I$  is

$$W^I = \int_0^{2\pi} \int_{\rho=0}^{\rho=b} \frac{1}{2} \operatorname{Re} E_\rho^I H_\phi^{I*} d\rho da \quad (6.8)$$



where  $da = \rho d\rho d\phi$ .

Substituting (5.28) and (5.27) into (6.8), we obtain

$$W^I = \left( \frac{\pi \zeta_o}{\omega \epsilon_1} \right) \left[ A_{II} \frac{K_1(\xi)}{J_1(x_1)} \right]^2 \int_{\rho=0}^{\rho=b} \rho J_1^2(x_1 \frac{\rho}{b}) d\rho. \quad (6.9)$$

The value of the definite integral in (6.9) is given by McLachlan,<sup>13</sup>

so (6.9) reduces to

$$W^I = \frac{1}{\epsilon_1} \left( \frac{\pi \zeta_o}{\omega \epsilon_o} \right) \left[ A_{II} \frac{K_1(\xi)}{J_1(x_1)} \right]^2 \frac{b^2}{2} \left\{ J_o^2(x_1) + J_1^2(x_1) - \frac{2}{x_1} J_o(x_1) J_1(x_1) \right\} \quad (6.10)$$

For region II where  $\rho \geq b$ ,  $W^{II}$  is given by

$$W^{II} = \int_0^{2\pi} \int_{\rho=b}^{\rho=\infty} \frac{1}{2} \operatorname{Re} E_{\rho}^{II} H_{\phi}^{II*} da. \quad (6.11)$$

Substituting (5.29) and (5.25) into (6.11) yields

$$W^{II} = \left( \frac{\pi \zeta_o}{\omega \epsilon_o} \right) A_{II}^2 \int_{\rho=b}^{\rho=\infty} \rho K_1^2(\xi \frac{\rho}{b}) d\rho \quad (6.12)$$

The definite integral in (6.12) is also given by McLachlan,<sup>14</sup> so we obtain  $W^{II}$  as

$$W^{II} = \left( \frac{\pi \zeta_o}{\omega \epsilon_o} \right) A_{II}^2 \frac{b^2}{2} \left\{ K_o^2(\xi) - K_1^2(\xi) + \frac{2}{\xi} K_o(\xi) K_1(\xi) \right\} \quad (6.13)$$

The constant  $A_{II}$  is defined in Eq. (5.25). We may now substitute

(6.10), (6.13), and the value of  $A_{II}$  into (6.7) to obtain  $W^S$ . After



considerable manipulation of constants one obtains the expression for the total surface wave power which is

$$W^S = \sqrt{\frac{\epsilon_o}{\mu_o}} \left( \frac{\epsilon_r a}{b} \right)^2 N^S \quad (6.14)$$

where

$$N^S = \pi \lambda_g / \lambda_o \left\{ \frac{\frac{1}{\epsilon_r} \left[ \frac{K_1(\xi)}{J_1(X_1)} \right]^2 \left[ J_o^2(X_1) + J_1^2(X_1) - \frac{2}{X_1} J_o(X_1) J_1(X_1) \right] +}{2 \left[ \frac{\left( \frac{X_1}{\xi} \right)^2 + 1}{X_1} \right] J_o(X_1) K_1(\xi) + \left[ \epsilon_r \left( \frac{\xi}{X_1} \right) + \left( \frac{X_1}{\xi} \right) \right] J_o(X_1) K_o(\xi) +}{+ \left[ K_o^2(\xi) - K_1^2(\xi) + \frac{2}{\xi} K_o(\xi) K_1(\xi) \right]} \right\} J_1^2 \left( \frac{k_o a}{k_o b} X_1 \right) + \left( \epsilon_r - 1 \right) J_1^2(X_1) K_1^2(\xi) \left\}^2$$

Inspection of the expression for  $N^S$  shows that the source parameter

$(k_o a)$  appears only in the argument of  $J_1^2 \left( \frac{k_o a}{k_o b} X_1 \right)$ . The remaining portion, which is rather formidable, is a function  $\epsilon_r$  and the mode solution  $(\xi, X_1)$ . It has a constant value for any  $\epsilon_r$  and  $k_o b$  which, of course, yields the eigenvalues  $\xi$  and  $X_1$ .  $N^S$  is a function of the

source dimension only by the term  $J_1^2 \left( \frac{k_o a}{k_o b} X_1 \right)$ . It should be noted,

however, that  $a/b$  appears in the constant which multiplies  $N^S$  to

give  $W^S$ . Selecting  $k_o b$  with  $\epsilon_r = 2.56$ ,  $N^S$  was calculated as a

function of  $k_o a$  using values of  $\lambda_g / \lambda_o$ ,  $\xi$ , and  $X_1$  from Table I. The

computed values of  $N^S$  are given in Table II.



### 6.3 Excitation Efficiency

The excitation efficiency  $\Upsilon$  for the magnetic current ring on a lossless dielectric rod is given by

$$\Upsilon = \frac{W^S}{W^S + W^R} \quad (6.2)$$

When (6.6) and (6.14) are substituted into (6.2) the constant  $\sqrt{\frac{\epsilon_o}{\mu_o}} \left(\frac{r_a}{b}\right)^2$  may be cancelled, so that (6.2) reduces to

$$\Upsilon = \frac{N^S}{N^S + N^R} \quad (6.15)$$

Excitation efficiency was calculated according to (6.15) using values of  $N^S$  and  $N^R$  obtained for the particular source dimension  $k_o a$  and rod parameter  $k_o b$ . Table II presents  $N^S$ ,  $N^R$ , and  $\Upsilon$  for each  $k_o a$  and  $k_o b$ . Figures 8 and 9 show curves of efficiency as a function of  $k_o a$  for the selected values of  $k_o b$ . The curves are graphs of the data presented in Table II. The analysis indicates that a magnetic current ring of the proper size will provide very efficient excitation of the  $E_{01}$  mode. Efficiencies greater than 90% are predicted for  $k_o a$  approximately equal to 2.6, which agrees very well with the rough estimate of optimum dimensions that was made in Section 6.1. Experimental measurements of efficiency were performed in the laboratory as a check on the foregoing analysis. The experimental work is described in Chapter 7.



TABLE II

| $k_o^b$  | $k_o^a$ | $N^S$  | $N^R$  | $\gamma$ |
|----------|---------|--------|--------|----------|
| 2.2<br>↓ | .2      | .0179  | .0673  | .2101    |
|          | .4      | .0684  | .2507  | .2144    |
|          | .6      | .1427  | .5010  | .2217    |
|          | .8      | .2278  | .7517  | .2326    |
|          | 1.0     | .3087  | .9379  | .2476    |
|          | 1.2     | .3719  | 1.0130 | .2685    |
|          | 1.4     | .4066  | .9621  | .2971    |
|          | 1.6     | .4077  | .8026  | .3369    |
|          | 1.8     | .3755  | .5784  | .3936    |
|          | 2.0     | .3163  | .3469  | .4769    |
|          | 2.2     | .2403  | .1604  | .5997    |
| 2.6<br>↓ | .2      | .0320  | .1162  | .2159    |
|          | .4      | .1228  | .4337  | .2207    |
|          | .6      | .2582  | .8659  | .2297    |
|          | .8      | .4171  | 1.2969 | .2433    |
|          | 1.0     | .5744  | 1.6129 | .2626    |
|          | 1.2     | .7064  | 1.7357 | .2892    |
|          | 1.4     | .7928  | 1.6380 | .3261    |
|          | 1.6     | .8214  | 1.3540 | .3776    |
|          | 1.8     | .7891  | .9625  | .4505    |
|          | 2.0     | .7020  | .5620  | .5554    |
|          | 2.2     | .5741  | .2448  | .7011    |
| 3.0<br>↓ | 2.4     | .4249  | .0692  | .8599    |
|          | 2.6     | .2764  | .0481  | .8518    |
|          | .2      | .0400  | .1169  | .2549    |
|          | .4      | .1544  | .4369  | .2611    |
|          | .6      | .3278  | .8753  | .2725    |
|          | .8      | .5370  | 1.3165 | .2879    |
|          | 1.0     | .7536  | 1.6489 | .3137    |
|          | 1.2     | .9493  | 1.7897 | .3466    |
|          | 1.4     | 1.0984 | 1.7099 | .3911    |
|          | 1.6     | 1.1817 | 1.4377 | .4511    |
|          | 1.8     | 1.1902 | 1.0474 | .5319    |
|          | 2.0     | 1.1234 | .6371  | .6381    |
|          | 2.2     | .9920  | .2988  | .7685    |
|          | 2.4     | .8136  | .0962  | .8943    |
|          | 2.6     | .6115  | .0481  | .9271    |
|          | 2.8     | .4109  | .1302  | .7594    |
|          | 3.0     | .2349  | .2859  | .4510    |



TABLE II (CONTINUED)

| $k_o b$ | $k_o a$ | $N^S$  | $N^R$  | $\gamma$ |
|---------|---------|--------|--------|----------|
| 3.4     | .2      | .0371  | .1189  | .2378    |
|         | .4      | .1442  | .4443  | .2450    |
|         | .6      | .3090  | .8917  | .2573    |
|         | .8      | .5130  | 1.3470 | .2758    |
|         | 1.0     | .7329  | 1.6951 | .3019    |
|         | 1.2     | .9438  | 1.8523 | .3375    |
|         | 1.4     | 1.1228 | 1.7866 | .3859    |
|         | 1.6     | 1.2501 | 1.5238 | .4507    |
|         | 1.8     | 1.3124 | 1.1342 | .5364    |
|         | 2.0     | 1.3038 | .7142  | .6461    |
|         | 2.2     | 1.2271 | .3575  | .7743    |
|         | 2.4     | 1.0913 | .1306  | .8931    |
|         | 2.6     | .9076  | .0583  | .9396    |
|         | 2.8     | .7087  | .1212  | .8540    |
|         | 3.0     | .5032  | .2667  | .6536    |
|         | 3.2     | .3164  | .4290  | .4244    |
|         | 3.4     | .1646  | .5487  | .2308    |
| 3.8     | .2      | .0334  | .1561  | .1763    |
|         | .4      | .1304  | .5843  | .1825    |
|         | .6      | .2816  | 1.1733 | .1936    |
|         | .8      | .4723  | 1.7725 | .2104    |
|         | 1.0     | .6845  | 2.2325 | .2347    |
|         | 1.2     | .8984  | 2.4437 | .2688    |
|         | 1.4     | 1.0933 | 2.3639 | .3162    |
|         | 1.6     | 1.2511 | 2.0228 | .3821    |
|         | 1.8     | 1.3583 | 1.5159 | .4726    |
|         | 2.0     | 1.4048 | .9668  | .5923    |
|         | 2.2     | 1.3873 | .4983  | .7357    |
|         | 2.4     | 1.3088 | .1967  | .8693    |
|         | 2.6     | 1.1772 | .0946  | .9255    |
|         | 2.8     | 1.0056 | .1674  | .8573    |
|         | 3.0     | .8096  | .3485  | .6991    |
|         | 3.2     | .6073  | .5530  | .5234    |
|         | 3.4     | .4154  | .7052  | .3767    |
|         | 3.6     | .2492  | .7584  | .2473    |
|         | 3.8     | .1209  | .7048  | .1464    |



TABLE II (CONTINUED)

| $k_o b$ | $k_o a$ | $N^S$  | $N^R$  | $\tau$ |
|---------|---------|--------|--------|--------|
| 4.2     | .2      | .0290  | .2710  | .0967  |
|         | .4      | .1138  | 1.0130 | .1010  |
|         | .6      | .2471  | 2.0307 | .1085  |
|         | .8      | .4184  | 3.0632 | .1202  |
|         | 1.0     | .6140  | 3.8423 | .1378  |
|         | 1.2     | .8177  | 4.1927 | .1632  |
|         | 1.4     | 1.0133 | 4.0300 | .2009  |
|         | 1.6     | 1.1854 | 3.4293 | .2569  |
|         | 1.8     | 1.3205 | 2.5469 | .3414  |
|         | 2.0     | 1.4326 | 1.6114 | .4706  |
|         | 2.2     | 1.4671 | .8292  | .6389  |
|         | 2.4     | 1.4211 | .3379  | .8079  |
|         | 2.6     | 1.3465 | .1901  | .8763  |
|         | 2.8     | 1.2259 | .3336  | .7861  |
|         | 3.0     | 1.0687 | .6461  | .6232  |
|         | 3.2     | .8876  | .9817  | .4748  |
|         | 3.4     | .6967  | 1.2140 | .3646  |
|         | 3.6     | .5096  | 1.2727 | .2859  |
|         | 3.8     | .3396  | 1.1546 | .2273  |
|         | 4.0     | .1973  | .9168  | .1771  |
|         | 4.2     | .0912  | .6461  | .1237  |



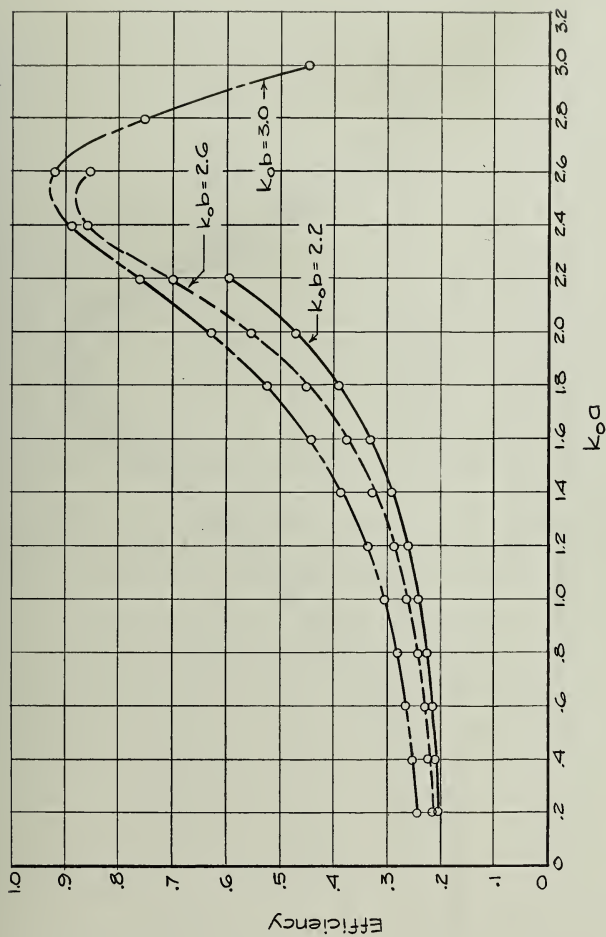


FIGURE 8. EXCITATION EFFICIENCY AS A FUNCTION OF THE SOURCE DIMENSION  $k_0 a$



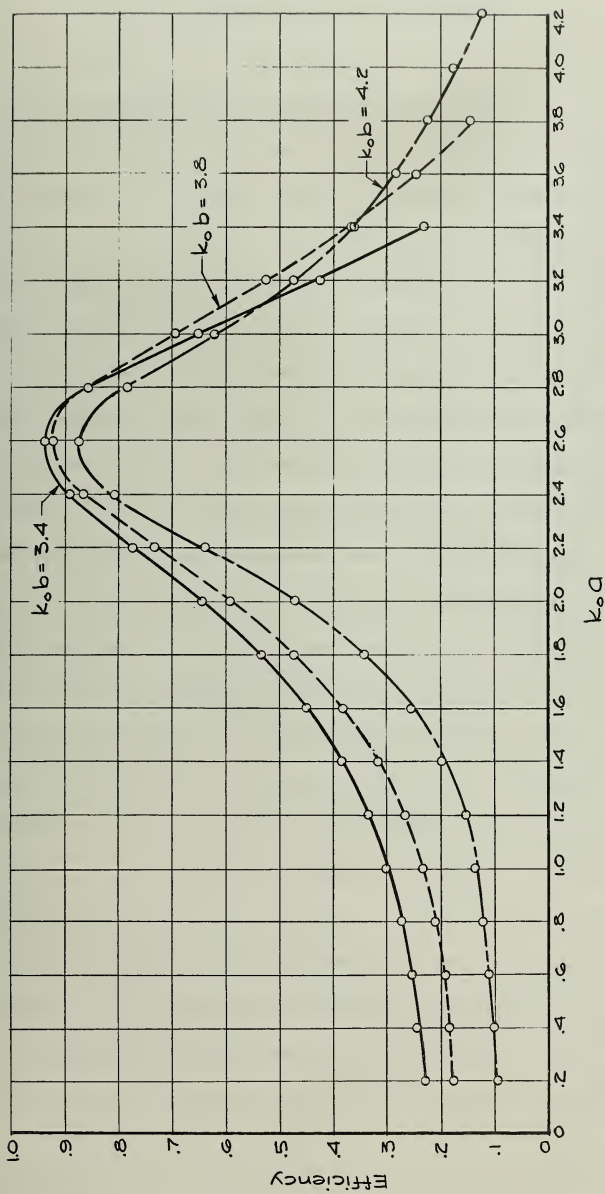


FIGURE 9. EXCITATION EFFICIENCY AS A FUNCTION OF THE SOURCE DIMENSION  $k_0 a$



## 7. EXPERIMENTAL INVESTIGATION

### 7.1 The Source Representation and Measurement Method

The magnetic current ring was purposely chosen for the theoretical problem because it is a source which has a simple physical realization. Consider Fig. 1. By symmetry the magnetic current filament generates a field which has no electric component in the plane  $z = 0$ . In the plane of the source the only non-zero component of  $\vec{E}$  is  $E_z$ , which is normal to the plane. Since tangential  $\vec{E}$  vanishes over the plane, an infinitely large metal sheet may be placed at the source position without disturbing the field. One may simulate the source by a very narrow annular slot in a large conducting sheet or ground plane. Then, of course, the dielectric rod extends from the ground plane in only one direction. The annular slot may be properly illuminated by a circular or coaxial waveguide. If the rod is terminated in a resistance card load which results in negligible reflection of the surface wave, then the finite rod excited by an annular slot in a ground plane simulates in a half space the theoretical problem of Fig. 1. A photograph of the experimental structure assembled in the laboratory is given in Fig. 10.

In order for the slot to be a good approximation of the filament, the slot width must be small compared to the wavelength and the electric field in the slot must be radial and uniform about the circumference. It is assumed that the mean radius of the slot corresponds to the radius of the infinitesimal current filament. The slot may be illuminated from a circular waveguide propagating the  $TM_{01}$  mode, or



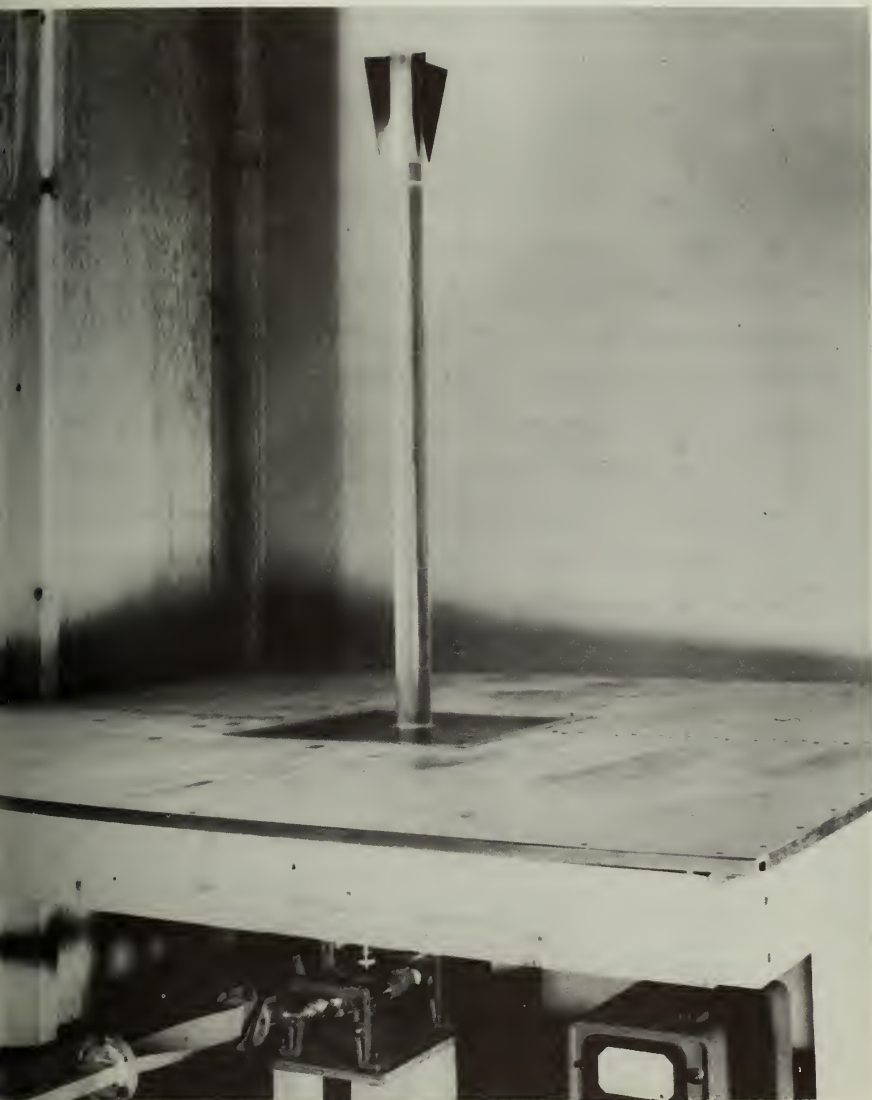


FIGURE 10. VIEW OF THE DIELECTRIC ROD AND GROUND PLANE



from a coaxial line excited in the usual TEM mode. The coaxial line is preferable since the center conductor provides a convenient means of supporting the circular disc which is the ground plane within the annular slot.

The efficiency with which the slot excites the surface wave may be measured using Deschamps' method<sup>5,6</sup> for determining the scattering matrix coefficients of a waveguide junction. The transition from the coaxial line to the dielectric rod waveguide is considered as a two port waveguide junction. Assuming that the dielectric rod and coaxial line are lossless, then the dissipative attenuation of the junction results entirely from the power radiated into space by the slot. If all of the power delivered to the slot is converted to the surface wave mode, then there is no power lost as stray radiation and the junction is 100% efficient. This method has been used previously to measure the launching efficiency of sources placed on a dielectric image line.<sup>15</sup>

The experimental procedure for measuring the efficiency of the junction is illustrated in Fig. 11.

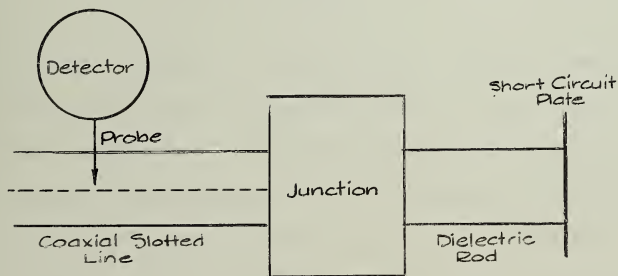


FIGURE 11. REPRESENTATION OF A TWO PORT JUNCTION



The short circuit plate which terminates the dielectric rod is placed at four arbitrary but equally spaced positions. The reflection coefficient corresponding to each position of the short is measured in the coaxial line and plotted on the reflection coefficient plane. The measured points define a circle which is the image of the unit circle in the output guide. Using the graphical constructions derived by Deschamps, one may determine the crossover point, iconocenter, and, ultimately, the scattering matrix coefficients which characterize the junction.

The graphical construction used to determine the iconocenter may be avoided if a matched load is available for the output guide, since the coefficient  $S_{11}$  and the iconocenter are the same point on the reflection coefficient plane. If one terminates the output guide in a matched load, the reflection coefficient measured in the input guide is  $S_{11}$  and, therefore, the iconocenter.

The so-called "matched" efficiency,  $\Upsilon$ , is defined as

$$\Upsilon = \frac{P_m}{P_i} = \frac{|S_{12}|^2}{1 - |S_{11}|^2} \quad (7.1)$$

where  $P_m$  is the power delivered to a matched load (infinitely long guide) and  $P_i$  is the power delivered to the junction when the output guide is terminated by a matched load. We see that  $\Upsilon$  is identical to the excitation efficiency defined by Eq. (6.1). The relation between  $\Upsilon$  and the scattering coefficients is given in Eq. (7.1). In order to calculate  $\Upsilon$  one needs only the iconocenter and the locus of the image circle. A single measurement for the matched load termination determines



the iconocenter, and three measurements for arbitrary positions of the short circuit plate define the image circle. Mathis<sup>16</sup> gives a convenient formula for  $\Upsilon$  which is equivalent to the relation (7.1).

His formula is

$$\Upsilon = \frac{2D_1 D_2}{(D_1 + D_2)(1 - D_3^2)} \quad (7.2)$$

where  $D_1$ ,  $D_2$ , and  $D_3$  are the lengths defined in Fig. 12. The origin of the reflection coefficient plane is the point 0, the iconocenter is  $0'$ , and the center of the image circle is C. Formula (7.2) was used to calculate the efficiency for all of the experimental measurements. Before the results are presented, a brief description of the experimental apparatus and coaxial exciter will be given.

## 7.2 Experimental Equipment

A two inch diameter polystyrene rod was selected for the experimental work. This choice followed from two main considerations. First, a two inch rod can stand on end without additional support. This method of vertical mounting avoids the usual problem of how to support a surface wave structure without disturbing the field. Second, it is easier to fabricate annular slots for a large rod than for a small rod. Table III lists the wavelength  $\lambda_o$ , frequency  $f$ , the ratio  $\lambda_g/\lambda_o$ , and  $\lambda_g$  for a two inch polystyrene rod, assuming  $\epsilon_r = 2.56$ .

Using Deschamps' method, one obtains the efficiency of the entire transition between the input reference plane and the output reference plane, that is, between the measuring probe and the short circuit termination. Since we are interested in measuring the dissipative



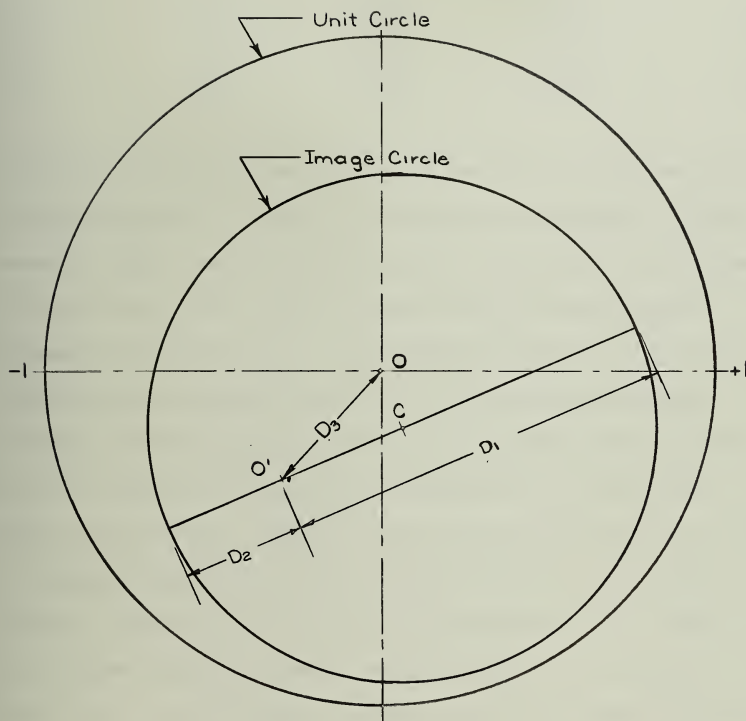


FIGURE 12. PLOT OF THE IMAGE CIRCLE ON THE REFLECTION COEFFICIENT PLANE



TABLE III

| $k_o b$ | $\lambda_o$<br>centimeters | $f$<br><u>megacycles</u><br>sec | $\lambda_g/\lambda_o$ | $\lambda_g$<br>centimeters |
|---------|----------------------------|---------------------------------|-----------------------|----------------------------|
| 2.2     | 7.25                       | 4132                            | 0.969                 | 7.03                       |
| 2.6     | 6.14                       | 4884                            | 0.904                 | 5.55                       |
| 3.0     | 5.32                       | 5635                            | 0.840                 | 4.47                       |
| 3.4     | 4.69                       | 6387                            | 0.791                 | 3.71                       |
| 3.8     | 4.20                       | 7138                            | 0.756                 | 3.17                       |
| 4.2     | 3.80                       | 7889                            | 0.730                 | 2.78                       |

attenuation of just the annular slot, it is essential that the rest of the transition shall introduce only negligible attenuation. For the present we shall ignore the dielectric rod loss since it is very small.

It follows, then, that a low loss exciter must be constructed to illuminate the annular slot. The annular slot presents a very low conductance and capacitive susceptance to the exciting waveguide. Therefore, the feed waveguide must be a low impedance line with some means of tuning out the capacitive susceptance of the slot at the ground plane position. Figure 13 shows a cross sectional view of the low impedance coaxial line which was constructed for this purpose. The inner diameter of the outer wall of the coax was  $1\frac{7}{8}$  inches and the inner conductor diameter was  $1\frac{1}{4}$  inches, which yields a 24 ohm line. A two wavelength tapered section transformed the 24 ohm line to a standard 50 ohm, type N connector. Two polystyrene rings centered the inner conductor within the cylinder. The ring located at the tapered section was one-half wavelength long, so that no impedance



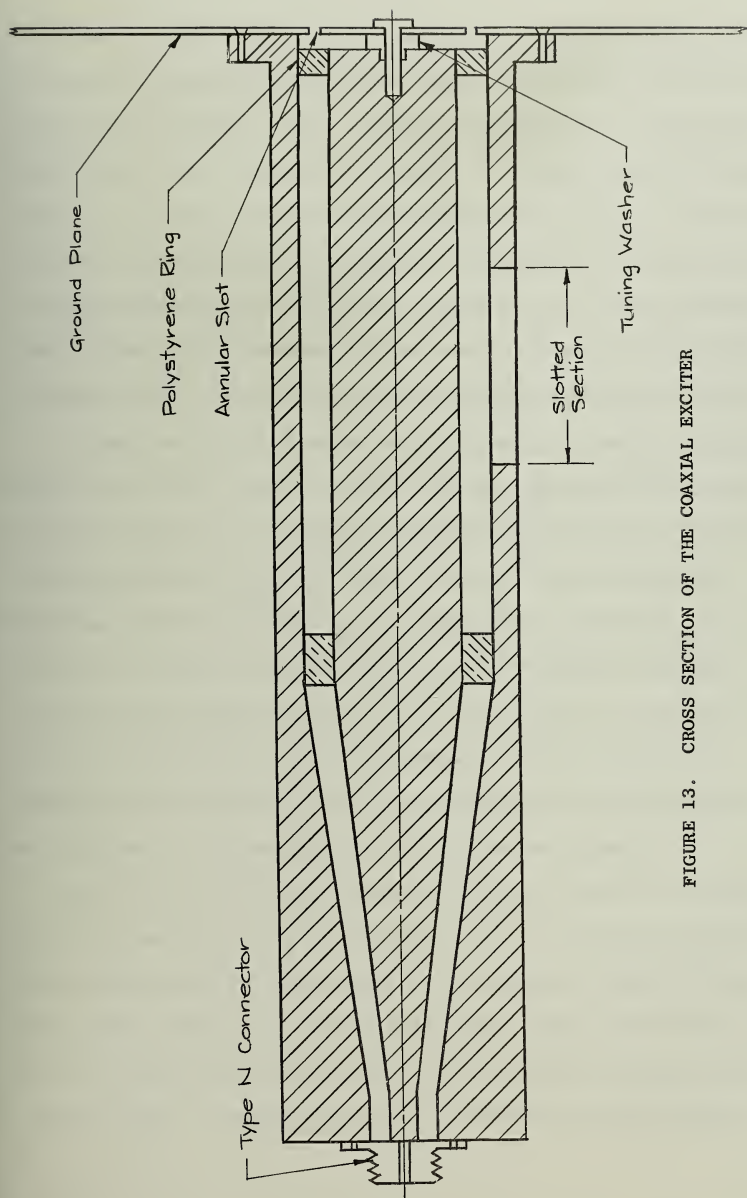


FIGURE 13. CROSS SECTION OF THE COAXIAL EXCITER



discontinuity would result. The ring at the ground plane position was a quarter-wavelength transformer which would match the 24 ohm line to a 9.5 ohm resistance load. A circular tuning disc or washer was placed on the end of the center conductor to provide a series inductive reactance to cancel the capacitive susceptance of the slot. The circular disc which formed the inner boundary of the annular slot was fixed to the tuning washer and the coax center conductor by a special mounting screw. Since the neighborhood of the annular slot and tuning washer is a resonant cavity, all parts were silver-plated to minimize losses.

A family of discs and rings were fabricated to permit varying the annular slot radius from  $1/2$  inch to  $7/8$  inch while maintaining the slot width constant at  $1/8$  inch. The  $1/8$  inch slot width corresponds to  $.067 \lambda_0$  and  $.076 \lambda_0$  at  $k_0 b$  equal to 3.4 and 3.8, respectively. Recalling that the radius of the dielectric rod is one inch, it is convenient to express the slot radius in the normalized form  $a/b$ ; then the source dimension  $k_0 a$  for any  $k_0 b$  is given by  $k_0 b (a/b)$ . Six slots were constructed for the measurements. Table IV gives the normalized slot radius  $a/b$ , and the corresponding dimension  $k_0 a$  at the two values of  $k_0 b$  which were used for the measurements.

A slot  $1/8$  inch wide and two inches long was milled in the side wall of the exciter so that the standing wave ratio could be measured. After a particular slot was mounted on the exciter, various tuning washers were tested until one was found which reduced the VSWR in the exciter to less than two when the dielectric rod was terminated with a matched load. The image circle was determined from measurements



TABLE IV

| Normalized Slot<br>Radius $a/b$ | $k_o a$<br>when $k_o b=3.4$ | $k_o a$<br>when $k_o b=3.8$ |
|---------------------------------|-----------------------------|-----------------------------|
| 0.50                            | 1.70                        | 1.90                        |
| 0.625                           | 2.12                        | 2.38                        |
| 0.687                           | 2.34                        | 2.61                        |
| 0.75                            | 2.55                        | 2.85                        |
| 0.812                           | 2.76                        | 3.08                        |
| 0.875                           | 2.98                        | 3.32                        |

made in a coaxial slotted line which was connected directly to the exciter through a type N elbow connector. A view of the slotted line, coaxial exciter, and the underside of the ground plane is given in Fig. 14. A block diagram of the waveguide apparatus used for the measurements is given in Fig. 15. When Deschamps' procedure was carried out, the dielectric rod was terminated with a six inch diameter plate, machined from 1/16 inch brass and silver plated. The ground plane, which may be seen in Fig. 10, was 60 inches square. For most of the measurements the dielectric rod was 40 centimeters long, although in some instances, the length was increased to 134 centimeters. Finally, it should be mentioned that an external probe was constructed to measure the standing wave ratio of the field along the dielectric rod. It is shown schematically in Fig. 15. It was used to measure the guide wavelength,  $\lambda_g$ , when the rod was operated at  $k_o b$  equal to 3.4 and 3.8. In each case, the measured guide wavelength corresponded exactly to the value predicted in Table III. A matched load was made from four tapered lengths of 300 ohm resistance cord which were



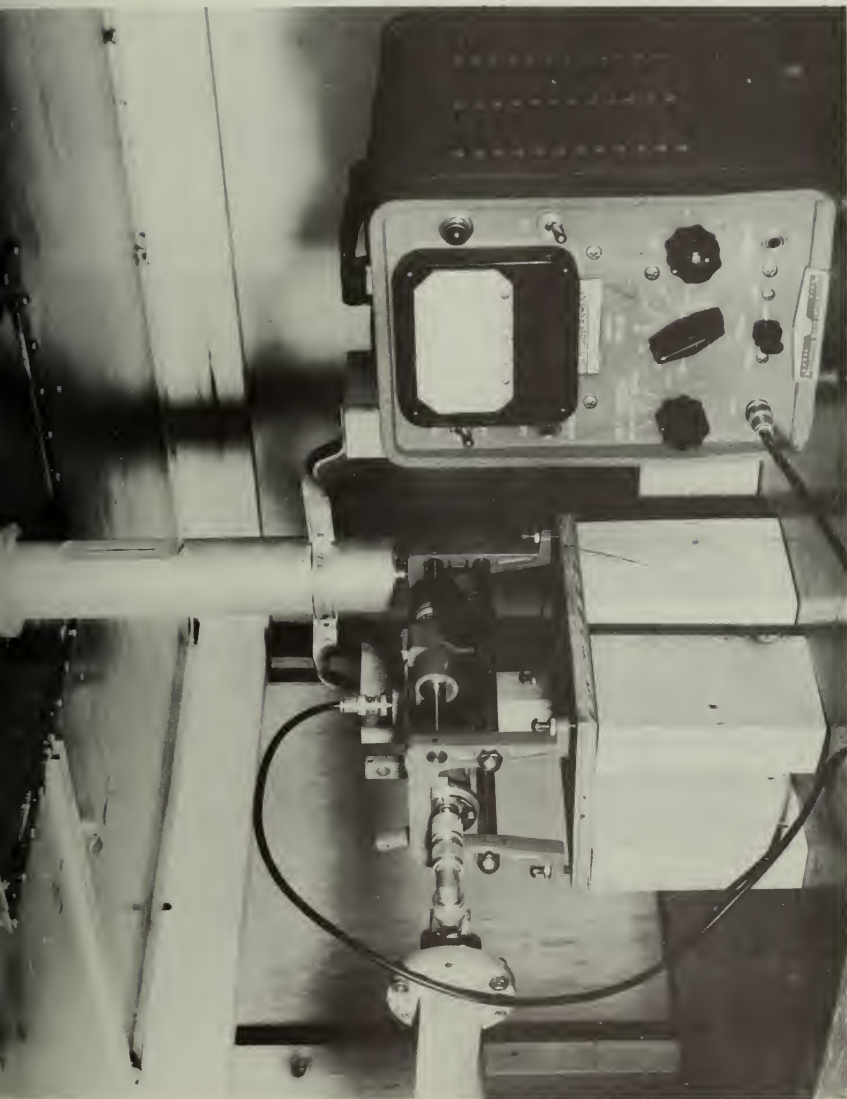


FIGURE 14. COAXIAL EXCITER MOUNTED ON THE GROUND PLANE  
(SEEN FROM BELOW)



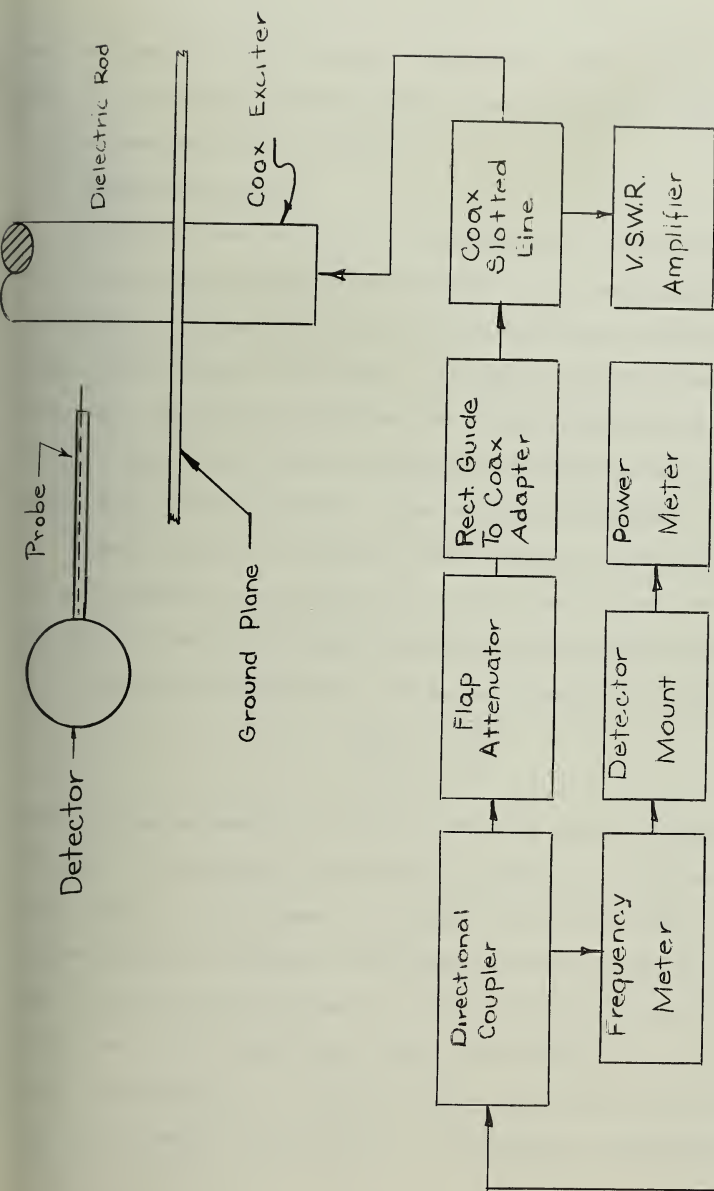


FIGURE 15. DIAGRAM OF THE MEASUREMENT EQUIPMENT



mounted symmetrically on the rod. The matched load is visible in Fig. 10. The VSWR on the dielectric rod was less than 1.1 when the rod was terminated with this load.

### 7.3 Experimental Results

When efficiency was measured as shown in Fig. 11, the result obtained was not precisely the excitation efficiency of the annular slot; instead it was the efficiency of the entire transition between the measuring probe and the short circuit plate. Although the losses in the system were small, the measured efficiency was reduced somewhat by dielectric rod attenuation and by the loss between the measuring probe and the ground plane. We shall discuss these two effects separately.

First, consider the dielectric rod attenuation which caused the two port network to be terminated in a lossy short circuit. The lossy rod acts as a reflection coefficient transformer, so that the magnitude of the reflection coefficient at the ground plane,  $|\Gamma|$ , is given by

$$|\Gamma| = e^{-2\alpha z} \quad (7.3)$$

where  $\alpha$  is the attenuation constant of the wave on the dielectric rod and  $z$  is the length of rod between the ground plane and the short circuit plate. It is assumed, of course, that a perfect short terminates the rod yielding a reflection coefficient of unity at that position. We wish to determine the efficiency of the slot while excluding the effect of the lossy rod. Consider Fig. 16(a), which shows  $[S']$ , the four pole to be measured; it is located between the slotted line and the lossy short circuit. Figure 16(b) is an alternate representation of



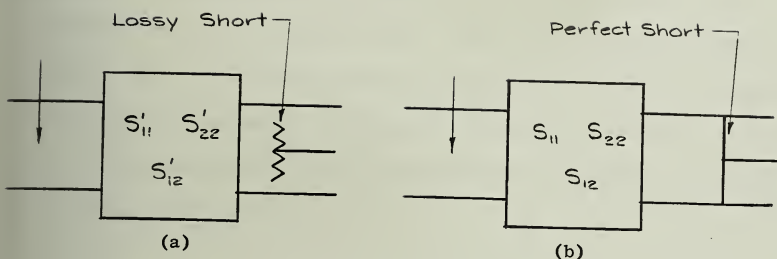


FIGURE 16. MEASUREMENT OF A JUNCTION WITH A LOSSY SHORT CIRCUIT

Fig. 16(a) in which the reflection coefficient transformer  $|\Gamma|$  has been associated with  $[S']$  to form the fictitious four pole  $[S]$  which is terminated with a perfect short. When one measures the four pole, assuming a perfect short, the composite four pole  $[S]$  is obtained. Assuming that  $|\Gamma|$  is known, the parameters of  $[S']$  are given by<sup>17</sup>

$$S'_{11} = S_{11}, \quad S'_{22} = \frac{S_{22}}{|\Gamma|}, \quad S'^2_{12} = \frac{S_{12}^2}{|\Gamma|} \quad (7.4)$$

Referring to Eq. (7.2), it is evident that the efficiency,  $\Upsilon'$ , of  $[S']$  is related to  $\Upsilon$  of  $[S]$  by

$$\Upsilon' = \frac{\Upsilon}{|\Gamma|^2} \quad (7.5)$$

$\Upsilon$  is the slot efficiency which would be measured if the dielectric rod were lossless, and Eq. (7.5) gives the relation between the measured



efficiency  $\Upsilon$ , and  $\Upsilon_i$ .

In order to apply (7.5) to the experimental measurements,  $|\Gamma|$  was obtained by measuring the efficiency of the same source with two different lengths of dielectric rod. The distance from the ground plane to the short circuit plate was 40 centimeters and 134 centimeters for the two measurements. Referring to Fig. 17, let  $\Upsilon$  and  $\Upsilon_o$  represent this efficiencies which would be measured when the short was located at the specified positions.

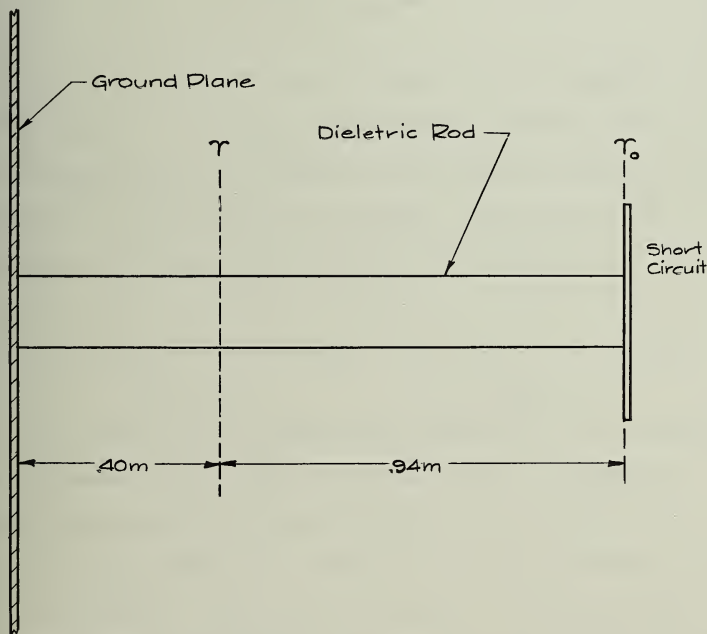


FIGURE 17. ARRANGEMENT FOR MEASURING ROD ATTENUATION



From (7.3) and (7.5) it follows that

$$\frac{\gamma}{\gamma_o} = e^{2az} \quad (7.6)$$

where  $z$  is equal to 0.94 meters.

Substituting the measured values of  $\gamma$  and  $\gamma_o$  into (7.6), one obtains the value of the attenuation constant  $a$ . Having determined  $a$ , it follows from (7.5) that

$$\gamma' = \gamma e^{2az} \quad (7.7)$$

where  $z$  is now 0.40 meters.  $\gamma'$  may be thought of as the measured efficiency referred to the ground plane position. Equation (7.7) defines the adjustment which is applied to the measured efficiency,  $\gamma$ , to account for the dielectric rod attenuation. The value of  $a$  was measured when  $k_o b$  was equal to 3.4 and 3.8. The results are

$$a \Big|_{k_o b = 3.4} = .0692, \quad a \Big|_{k_o b = 3.8} = .0716 \quad (7.8)$$

Before (7.7) is applied to the experimental measurements, the attenuation between the measuring probe and the ground plane will be discussed.

The attenuation of the transition between the measuring probe and the ground plane results from losses in the type N connector and the coaxial exciter. The loss is, of course, a function of the standing wave ratio in the transition. When a particular slot is being measured, the VSWR changes with the short circuit position because the junction is lossy. Therefore, the transition loss is different for



each position of the short, and it is impracticable to correct every standing wave measurement performed. One may, however, establish the maximum loss, or, correspondingly, the minimum efficiency of the transition. This was accomplished by mounting a two inch diameter, circular waveguide perpendicular to the ground plane and concentric to the annular slot, and measuring the efficiency by Deschamps' method. The circular guide was fitted with a movable shorting plunger. The annular slot excites the  $TM_{01}$  mode in the guide. The schematic representation of the two port junction is shown in Fig. 18.

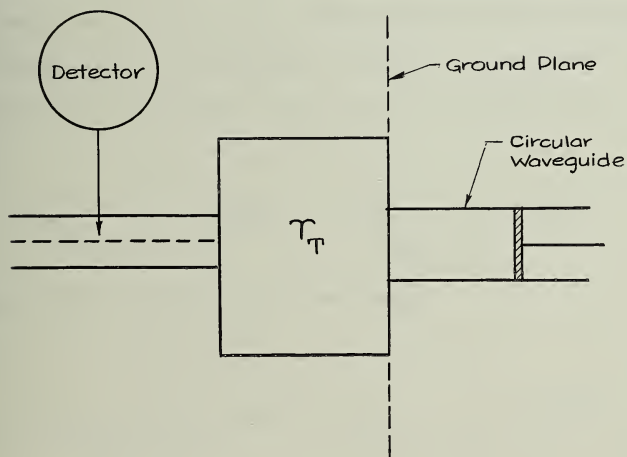


FIGURE 18. ARRANGEMENT FOR MEASURING THE TRANSITION EFFICIENCY



Assuming that the circular guide was lossless, the measured efficiency,  $\Upsilon_T$ , is the efficiency of the transition from the slotted line to the annular slot. It is now possible to adjust the measured slot efficiency to account for both the transition loss and the dielectric rod attenuation. Let  $\Upsilon_c$  denote the corrected slot efficiency; it is given by

$$\Upsilon_c = \frac{\Upsilon'}{\Upsilon_T} = \frac{\Upsilon_e^{2az}}{\Upsilon_T} \quad (7.9)$$

where  $\Upsilon$  is the efficiency measured with the lossy rod present, and  $\Upsilon_T$  is the transition efficiency. It should be noted that  $\Upsilon_T$  provides a maximum correction to  $\Upsilon$  since, in most cases, the transition efficiency will be greater than the value  $\Upsilon_T$ . This results because standing wave ratios are higher when  $\Upsilon_T$  is measured than when  $\Upsilon$  is determined.  $\Upsilon_T$  was measured when  $k_0 b$  was equal to 3.4 and 3.8. The results are

$$\Upsilon_T \bigg|_{k_0 b = 3.4} = 0.939, \quad \Upsilon_T \bigg|_{k_0 b = 3.8} = 0.934 \quad (7.10)$$

Substituting (7.8) and (7.10) into (7.9) yields

$$\Upsilon_c \bigg|_{k_0 b = 3.4} = 1.125 \Upsilon, \quad \Upsilon_c \bigg|_{k_0 b = 3.8} = 1.132 \Upsilon \quad (7.11)$$

The measured efficiencies were corrected according to Eqs. (7.11). Note that the correction factor is approximate. It was obtained in order to fix the order of magnitude of the system losses. One may see, however, that the excitation efficiency of a slot is somewhat



greater than the measured value.

Six slots were measured in the laboratory; the results are presented in Table V, which includes the measured efficiency  $\Upsilon$  and the corrected efficiency  $\Upsilon_c$  for each source  $k_o a$ . Measurements were performed at frequencies of 6387 megacycles/sec and 7138 megacycles/sec which result in  $k_o b$  equal to 3.4 and 3.8, respectively. The data of Table V have been plotted in Figs. 19 and 20 for comparison with the theoretical curves of efficiency. The very close agreement between the experimental points and the theoretical curve is evident. In all cases the measured

TABLE V

| $k_o b$  | $k_o a$ | Measured<br>Efficiency<br>$\Upsilon$ | Corrected<br>Efficiency<br>$\Upsilon_c$ |
|----------|---------|--------------------------------------|-----------------------------------------|
| 3.4<br>↓ | 1.70    | 0.44                                 | 0.495                                   |
|          | 2.12    | 0.65                                 | 0.73                                    |
|          | 2.34    | 0.76                                 | 0.855                                   |
|          | 2.55    | 0.835                                | 0.94                                    |
|          | 2.76    | 0.77                                 | 0.867                                   |
|          | 2.98    | 0.65                                 | 0.732                                   |
| 3.8<br>↓ | 1.90    | 0.48                                 | 0.544                                   |
|          | 2.38    | 0.775                                | 0.88                                    |
|          | 2.61    | 0.85                                 | 0.96                                    |
|          | 2.85    | 0.77                                 | 0.872                                   |
|          | 3.08    | 0.63                                 | 0.714                                   |
|          | 3.32    | 0.44                                 | 0.498                                   |

efficiency, which included the system losses, was within 10 percent of the efficiency predicted by theory. The experimental measurements verify that an excitation efficiency of approximately 95 percent may be obtained from an annular slot of dimension  $k_o a = 2.6$ .



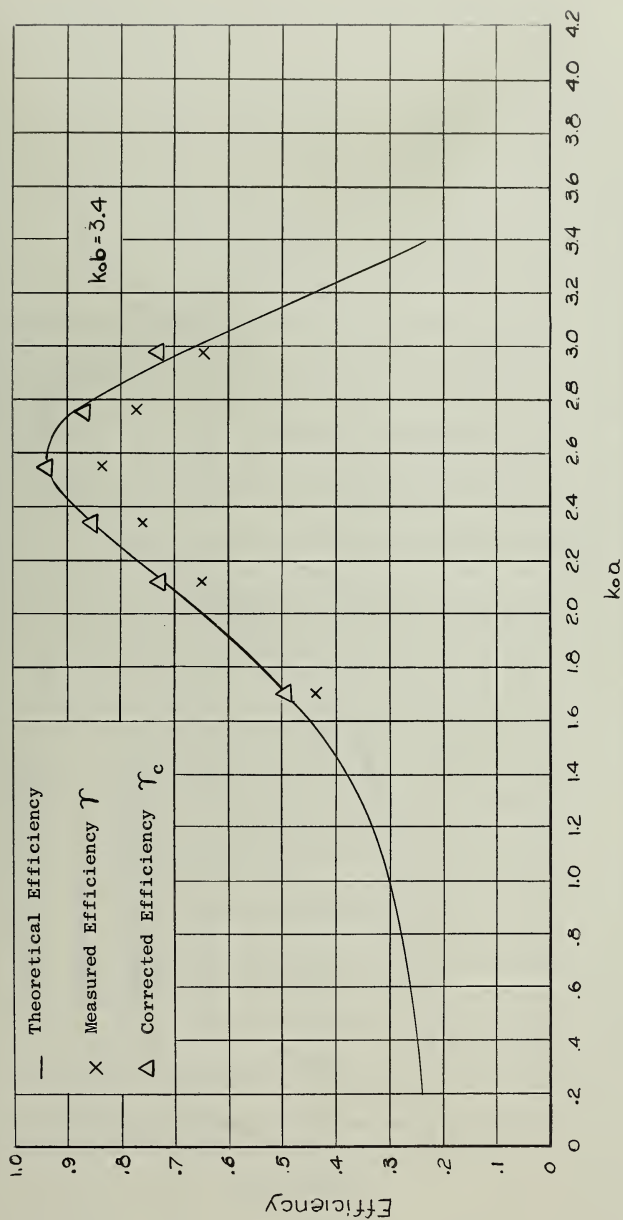


FIGURE 19. COMPARISON BETWEEN THEORETICAL AND MEASURED EFFICIENCY



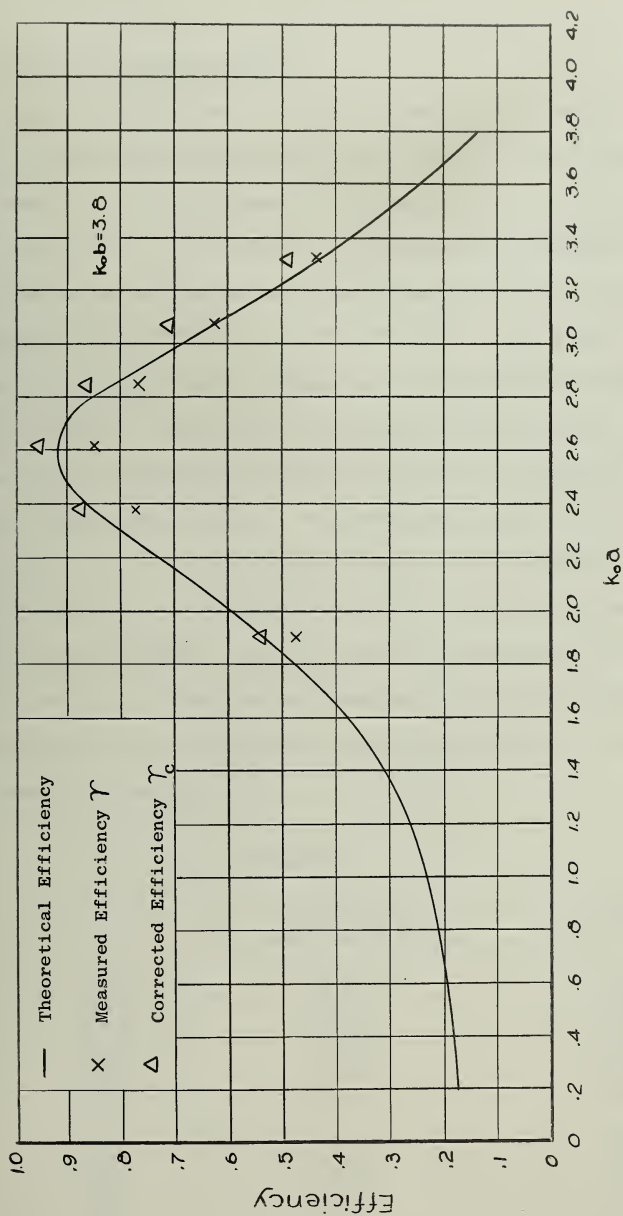


FIGURE 20 COMPARISON BETWEEN THEORETICAL AND MEASURED EFFICIENCY



#### 7.4 Elimination of the Ground Plane

The large ground plane was necessary in order for the annular slot to properly represent the theoretical source. A point of practical interest, however, is the efficiency of the annular slot in the presence of a small ground plane. This effect was investigated by measuring slot efficiencies as the size of the ground plane was reduced. Surprisingly, the ground plane had little effect on the efficiency. The ground plane was reduced to a 10 inch diameter flange and no appreciable change in efficiency was noted. Consequently, the ground plane was eliminated entirely so that the experimental structure consisted of the dielectric rod mounted on the coaxial exciter. The efficiency of each slot was measured at  $k_o b$  equal to 3.4 and 3.8. The results are presented in Table VI which includes, for comparison, the efficiency that was measured with the large ground plane. It may be seen that elimination of the ground plane had little effect on the launching efficiency of this source. The annular slot in the end of the coaxial section is a very efficient exciter of the  $E_{01}$  mode.

TABLE IV

| $k_o b$       | $k_o b$ | Efficiency $\Upsilon$<br>with Ground Plane | Efficiency $\Upsilon$<br>No Ground Plane |
|---------------|---------|--------------------------------------------|------------------------------------------|
| 3.4<br>↓<br>↓ | 1.70    | 0.44                                       | 0.428                                    |
|               | 2.12    | 0.65                                       | 0.647                                    |
|               | 2.34    | 0.76                                       | 0.779                                    |
|               | 2.55    | 0.835                                      | 0.847                                    |
|               | 2.76    | 0.77                                       | 0.829                                    |
|               | 2.98    | 0.65                                       | 0.696                                    |
| 3.8<br>↓<br>↓ | 1.90    | 0.48                                       | 0.545                                    |
|               | 2.38    | 0.775                                      | 0.776                                    |
|               | 2.61    | 0.85                                       | 0.835                                    |
|               | 2.85    | 0.77                                       | 0.776                                    |
|               | 3.08    | 0.63                                       | 0.63                                     |
|               | 3.32    | 0.44                                       | 0.413                                    |



## CONCLUSIONS

The theoretical analysis predicted that a circular filament 0.827 wavelengths in diameter would launch the  $E_{01}$  mode with an efficiency of approximately 95 percent. The analysis also showed that excitation efficiency is a critical function of the source dimension, and high efficiency is obtained only for a limited range of source diameters. It is evident that a filament which is greater in diameter than the dielectric rod is a poor exciter of the surface wave.

The uniformly illuminated annular slot was an excellent physical representation of the theoretical source. The measured slot launching efficiencies were in close agreement with the results predicted by the theory. It was determined that the annular slot efficiency is essentially independent of the size of the ground plane. Very good efficiency may be obtained using a simple launcher such as the coaxial exciter.

In order to realize the high efficiency of the annular slot, it is necessary to illuminate the slot from a good "matched" transition. In the laboratory, the predicted source efficiencies were not measured until the matched transition of the coaxial exciter was constructed. Of course, the efficiency of the source is independent of the feeding structure, but substantial coupling of power from the closed waveguide to the surface waveguide is obtained only when a good impedance match is provided at the aperture plane. Otherwise, a very high standing wave ratio exists in the vicinity of the aperture and most of the power delivered by the feeding waveguide is dissipated as loss at this



position. A matched transition is of utmost importance when one wishes to excite any type of surface waveguide.

Finally, good experimental verification of the pole, saddle point method of calculating efficiency was obtained.



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## APPENDIX A

In order to solve for the constant  $D$ , we write the system of equations (3.12) in matrix form:

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & 0 \\ 0 & \Delta_{32} & \Delta_{33} & \Delta_{34} \\ 0 & \Delta_{42} & \Delta_{43} & \Delta_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -i\omega\epsilon_1 b \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (A.1)$$

where

$$\begin{aligned} \Delta_{11} &= -(\nu_1 b) J_1'(\nu_1 a) & \Delta_{32} &= J_1(\nu_1 b) \\ \Delta_{12} &= -\Delta_{11} & \Delta_{33} &= Y_1(\nu_1 b) \\ \Delta_{13} &= (\nu_1 b) Y_1'(\nu_1 a) & \Delta_{34} &= -H_1^{(1)}(\nu_0 b) \\ \Delta_{21} &= J_1(\nu_1 a) & \Delta_{42} &= (\nu_1 b) J_1'(\nu_1 b) \\ \Delta_{22} &= -\Delta_{21} & \Delta_{43} &= (\nu_1 b) Y_1'(\nu_1 b) \\ \Delta_{23} &= -Y_1(\nu_1 a) & \Delta_{44} &= \epsilon_r(\nu_0 b) H_1^{(1)}(\nu_0 b) + (\epsilon_r - 1) H_1^{(1)}(\nu_0 b) \end{aligned}$$

Denote the coefficient matrix of (A.1) by  $[\Delta_{ij}]$  and let the cofactor of  $\Delta_{ij}$  be designated  $\lambda_{ij}$ . The solution of (A.1) is

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \frac{1}{\text{Det} [\Delta_{ij}]} \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} & \lambda_{41} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} & \lambda_{42} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \lambda_{43} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} \end{bmatrix} \begin{bmatrix} -i\omega\epsilon_1 b \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (A.2)$$



which reduces to

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \frac{-i\omega\epsilon_1 b}{\text{Det} [\Delta_{1j}]} \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix} \quad (\text{A.3})$$

Evaluating  $\text{Det} [\Delta_{1j}]$ , we obtain

$$\begin{aligned} \text{Det} [\Delta_{1j}] &= \Delta_{11} (\Delta_{22} \Delta_{23} \Delta_{44} + \Delta_{23} \Delta_{34} \Delta_{42}) \\ &\quad - \Delta_{11} (\Delta_{32} \Delta_{23} \Delta_{44} + \Delta_{22} \Delta_{34} \Delta_{43}) \\ &\quad - \Delta_{12} (\Delta_{21} \Delta_{33} \Delta_{44} - \Delta_{21} \Delta_{34} \Delta_{43}) \\ &\quad + \Delta_{13} (\Delta_{21} \Delta_{32} \Delta_{44} - \Delta_{21} \Delta_{34} \Delta_{42}) \end{aligned} \quad (\text{A.4})$$

since  $\Delta_{11} = -\Delta_{12}$  and  $\Delta_{22} = -\Delta_{21}$ , (A.4) reduces to

$$\text{Det} [\Delta_{1j}] = (\Delta_{13} \Delta_{21} - \Delta_{11} \Delta_{23}) (\Delta_{32} \Delta_{44} - \Delta_{34} \Delta_{42}). \quad (\text{A.5})$$

We note that

$$\begin{aligned} (\Delta_{13} \Delta_{21} - \Delta_{11} \Delta_{23}) &= (\nu_1 b) [J_1(\nu_1 a) Y_1'(\nu_1 a) - Y_1(\nu_1 a) J_1'(\nu_1 a)] \\ &= (\nu_1 b) \frac{2}{\pi(\nu_1 a)} \end{aligned} \quad (\text{A.6})$$

since the bracketed term is the Wronskian of  $J_1(\nu_1 a)$ ,  $Y_1(\nu_1 a)$  which is equal to  $[2/\pi (\nu_1 a)]$ . Substituting (A.6) into (A.5) we obtain

$$\text{Det} [\Delta_{1j}] = \frac{2b}{\pi a} (\Delta_{32} \Delta_{44} - \Delta_{34} \Delta_{42}) \quad (\text{A.7})$$



The cofactor  $\lambda_{14}$  is given by

$$\lambda_{14} = -\Delta_{21} (\Delta_{32} \Delta_{43} - \Delta_{42} \Delta_{33}), \quad (\text{A.8})$$

From (A.3), (A.7) and (A.8) we have

$$D = -i\omega\epsilon_1 b \frac{\lambda_{14}}{\text{Det} [\Delta_{ij}]} = +i\omega\epsilon_1 a \frac{\Delta_{21} (\Delta_{32} \Delta_{43} - \Delta_{42} \Delta_{33})}{\frac{2}{\pi} (\Delta_{32} \Delta_{44} - \Delta_{34} \Delta_{42})}. \quad (\text{A.9})$$

Consider

$$\begin{aligned} (\Delta_{32} \Delta_{43} - \Delta_{42} \Delta_{33}) &= (\nu_1 b) [J_1(\nu_1 b) Y_1'(\nu_1 b) - Y_1(\nu_1 b) J_1'(\nu_1 b)] \\ &= (\nu_1 b) \frac{2}{\pi(\nu_1 b)} = \frac{2}{\pi} \end{aligned} \quad (\text{A.10})$$

because of Wronskian relation. Utilizing Bessel function recurrence relations we determine that

$$(\Delta_{32} \Delta_{44} - \Delta_{34} \Delta_{42}) = (\nu_1 b) J_0(\nu_1 b) H_1^{(1)}(\nu_0 b) - \epsilon_r(\nu_0 b) J_1(\nu_1 b) H_0^{(1)}(\nu_0 b) \quad (\text{A.11})$$

Substituting (A.10) and (A.11) into (A.9) yields

$$D = \frac{+i\omega\epsilon_1 a J_1(\nu_1 a)}{(\nu_1 b) J_0(\nu_1 b) H_1^{(1)}(\nu_0 b) - \epsilon_r(\nu_0 b) J_1(\nu_1 b) H_0^{(1)}(\nu_0 b)} \quad (\text{A.12})$$



## APPENDIX B

The graphical solution of Eqs. (4.9) and (4.10) is illustrated in Fig. 4 of Chapter 4. Equations (4.9) and (4.10) are

$$-X \frac{J_0(X)}{J_1(X)} - \varepsilon_r(\xi) \frac{K_0(\xi)}{K_1(\xi)} = 0 \quad (4.9)$$

$$X^2 + \xi^2 = (k_0 b)^2 (\varepsilon_r - 1) = R^2 \quad (4.10)$$

where we omit the subscript 1 of  $X$  for convenience of notation.

Referring to Fig. 4, we see that the intersection of the circle and the first branch of  $X = f(\xi)$  is the unique solution of (4.9) and (4.10). Consider the magnified view at the point of intersection of the curves, as shown in Fig. 21.

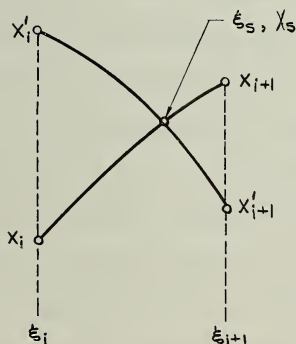


FIGURE 21. SOLUTION OF THE MODE EQUATION



Denote points on the circle corresponding to values of  $\xi$  by  $X'$ , where  $X' = \sqrt{R^2 - \xi^2}$  from (4.10). The points  $X$ , which define the curve with positive slope, are solutions of (4.9), which may be written in the form

$$-X \frac{J_0(X)}{J_1(X)} - \gamma = 0 \quad (\text{B.1})$$

where  $\gamma = \varepsilon_r(\xi) \frac{K_0(\xi)}{K_1(\xi)}$  is a constant for any selected  $\varepsilon_r$  and value of  $\xi$ .

We see that the points  $X$  are simply real-roots of the equation  $-X \frac{J_0(X)}{J_1(X)} - \gamma = 0$ . Let the point of intersection, which is the exact solution of (4.9) and (4.10), have the coordinates  $\xi_s$  and  $X_s$ . The basic method for determining the point of intersection using the digital computer is illustrated in Fig. 21. It is evident that for any value  $\xi$  less than  $\xi_s$ ,  $X' > X$ , and when  $\xi$  is greater than  $\xi_s$ ,  $X' < X$ . By computing  $X'$  and  $X$  for sufficiently small increments of  $\xi$ , it is possible to determine the neighborhood of intersection very accurately.

Two programs were written to determine  $\xi_s$  and  $X_s$  to an accuracy of four decimal places. The British Association Mathematical Tables<sup>18</sup> tabulate the functions  $K_0(\xi)$  and  $K_1(\xi)$  for .01 increments of  $\xi$ . The first program used the tabulated functions to determine the intersection as occurring between  $\xi_i$  and  $\xi_{i+1}$ , where the interval  $\xi_{i+1} - \xi_i = 0.01$ . The second program included an interpolation routine which subdivided the previously determined intersection interval into 100 parts and thereby yielded  $\xi_s$  and  $X_s$  accurate to four decimal places. A description of each program is given in Sections B.1 and B.2, respectively.



### B.1

The ILLIAC registers are limited to numbers in the range  $-1 \leq p < 1$ . All numerical computations must yield results which are less than one in absolute value; otherwise the registers overflow and one obtains meaningless results. Fractions read into the computer must also be in the range  $-1 \leq p < 1$ . Scaling of computations is almost always necessary and this program was no exception. The program was written using the scaled quantities  $k_0 b/8$ ,  $\mathcal{E}_r/4$ ,  $\xi/4$  and  $X/4$ . If the functions  $K_0(\xi)$  and  $K_1(\xi)$  exceeded unity, the values actually read into the computer were  $1/2 K_0(\xi)$  and  $1/2 K_1(\xi)$ . Selecting  $\mathcal{E}_r = 2.56$  and a particular value of  $k_0 b/8$ , the routine computed ten values of  $\xi/4$ ,  $X/4$ , and  $X'/4$  for .01 increments of  $\xi$  in the vicinity of the point of intersection. The program utilized the following ILLIAC library routines:

- H1 - Inverse Interpolation; finds a real root of  $f(X) = 0$  between two values  $X_1$  and  $X_2$  of the argument of  $f(X)$ .
- N12 - Infraput; a routine to read in up to 12 digit fractions or integers which are punched on tape. The sequence of fractions or integers is placed in memory locations  $n$ ,  $n+1$ ,  $n+2$ , ....
- P16 - Infraprint; this integer, fraction print routine prints the contents of the accumulator up to 12 decimal places, with optional choice of decimal point, sign, etc.
- R1 - Square Root; uses Newton's method to replace  $p$  in the accumulator by  $\sqrt{p}$ .
- V8 - Ordinary Bessel Functions; computes  $J_0(X)$ ,  $J_1(X)$ ,  $Y_0(X)$ , and  $Y_1(X)$  to an error of less than  $2^{-16}$ , using series expansions for the functions.



Library routine H1 requires the user to write an auxiliary subroutine which computes  $f(X)$ . The subroutine which was written refers to V8 in order to compute  $f(X) = -X \frac{J_0(X)}{J_1(X)} - \gamma$ .

The program commenced by reading in ten values of  $K_0(\xi)$  and placing them in memory locations 500 through 509; ten corresponding values of  $K_1(\xi)$  were placed in locations 510 through 519, while placing initial  $\xi/4$  in 520,  $X_1/4$  in 521,  $X_2/4$  in 522, and  $k_0b/8$  in 523. Note that  $X_1$  and  $X_2$  are arbitrary bounds of  $X$  within which a root of  $f(X) = 0$  is assumed to occur. As an example, consider the case  $k_0b = 3.0$ , which yields  $R = 3.75$ . Figure 4 indicates the intersection at approximately  $\xi = 1.97$ ,  $X = 3.2$ , so  $\xi$  was chosen to have the values  $\xi = 1.92, 1.93, 1.94, \dots, 2.01$  with  $X_1 = 3.16$  and  $X_2 = 3.26$ . Starting with the initial value of  $\xi/4$ ,  $1/16\gamma = \frac{\epsilon_r}{4} \frac{\xi}{4} \frac{K_0(\xi)}{K_1(\xi)}$  was computed, and library routine H1 was entered to obtain a scaled root  $X/4$  of the scaled equation

$$-\frac{1}{16}(X) \frac{J_0(X)}{J_1(X)} - \frac{1}{16}\gamma = 0. \text{ Using } k_0b/8, \epsilon_r/4, \text{ and } \xi/4, X^*/4 \text{ was}$$

calculated from the equation  $X^*/4 = \frac{1}{4} \sqrt{(k_0b)^2 (\epsilon_r - 1) - \xi^2}$ . The quantities  $k_0b/8$ ,  $\xi/4$ ,  $X/4$ , and  $X^*/4$  were printed to four decimal places,  $\xi$  was increased by .01 and the computation was repeated. The cycle continued until all ten values of  $\xi$  had been used, which completed the computations for that particular  $k_0b$ . New constants for a different  $k_0b$  were read from the tape and the entire routine was repeated. A first approximation of  $\xi_s$  and  $X_s$  for the six values of  $k_0b$  was obtained in this manner. The results are presented in Table B.1. Preceding each set of data is the value of  $k_0b/8$ . The first, second, and third columns give  $\xi/4$ ,  $X/4$ , and  $X^*/4$ ,



respectively. The arrow indicates the interval of  $\xi$  within which inter-  
section occurs. The tape format and order codes of the routines are  
as follows:

### Tape Format

|       |    |     |          |                                         |
|-------|----|-----|----------|-----------------------------------------|
|       | 00 | 20  | K        |                                         |
| (N12) | 00 |     | K        | Routine N 12                            |
| (P16) | 00 |     | K        | Routine P 16                            |
|       | 00 | 3   | K        | } pre-set parameter for H1              |
|       | 00 | F   | 00 600 F |                                         |
|       | 00 | 115 | K        | Routine H1                              |
| (R1)  | 00 |     | K        | Routine R1                              |
|       | 00 |     | K        | Routine V8                              |
|       | 00 | 600 | K        | Auxiliary Subroutine                    |
|       | 00 | 525 | K        | Main Routine                            |
|       | 24 | 525 | N        | Stop - Transfer control to main routine |

### Main Routine

|   |      |       |                                                     |
|---|------|-------|-----------------------------------------------------|
| 0 | → 50 | 500 F | ] read in 24 fractions                              |
|   | 50   | L     |                                                     |
| 1 | 26   | 20 F  |                                                     |
|   | L5   | 520 F | ] form "end" test constant $\frac{\xi_0 + .088}{4}$ |
| 2 | L4   | (8)   |                                                     |
|   | 40   | (9)   |                                                     |
| 3 | 92   | 707 F |                                                     |
|   | 92   | 131 F |                                                     |
| 4 | 92   | 515 F |                                                     |



|     |    |       |                                        |
|-----|----|-------|----------------------------------------|
|     | L5 | 523 F |                                        |
| 5   | J4 | 3 F   | print $k_0^{b/8}$                      |
|     | 50 | 5 L   |                                        |
| 6   | 26 | 59 F  |                                        |
|     | L5 | 520 F |                                        |
| 7   | 40 | (1)   | initial $\xi/4$ in (1)                 |
| →   | 92 | 131 F |                                        |
| 8   | 92 | 515 F |                                        |
|     | L5 | (1)   |                                        |
| 9   | J4 | 4 F   | print $\xi_i/4$                        |
|     | 50 | 9 L   |                                        |
| 10  | 26 | 59 F  |                                        |
|     | 92 | 975 F | 4 spaces                               |
| 11* | 50 | (1)   |                                        |
|     | 7J | 500 F |                                        |
| 12* | 50 | (2)   | $\frac{\xi_i}{4} \frac{K_0}{K_1}$ in Q |
|     | 66 | 510 F |                                        |
| 13  | 7J | (3)   |                                        |
|     | 40 | (4)   | $\frac{1}{16} \gamma$ in (4)           |
| 14  | L5 | 521 F |                                        |
|     | 40 | 10 F  | $\frac{1}{4} X_1$ in 10                |
| 15  | L5 | 522 F |                                        |
|     | 40 | 12 F  | $\frac{1}{4} X_2$ in 12                |



|    |    |       |                                                        |
|----|----|-------|--------------------------------------------------------|
| 16 | J0 | (2)   | waste                                                  |
|    | 50 | 16L   | enter H1                                               |
| 17 | 26 | 115 F |                                                        |
|    | J0 | (2)   | waste                                                  |
| 18 | J4 | 4 F   |                                                        |
|    | 50 | 18 L  | print $X_1/4$                                          |
| 19 | 26 | 59 F  |                                                        |
|    | 92 | 975 F | 4 spaces                                               |
| 20 | L5 | (3)   |                                                        |
|    | L0 | (5)   | $\frac{1}{4} (\xi_v - 1)$ in (6)                       |
| 21 | 40 | (6)   |                                                        |
|    | 50 | 523 F |                                                        |
| 22 | 7J | 523 F | $(kb/8)^2$ in (7)                                      |
|    | 40 | (7)   |                                                        |
| 23 | 50 | (6)   |                                                        |
|    | 7J | (7)   | $(\frac{1}{16})^2 [(kb)^2 (\xi_v - 1)]$ in (7)         |
| 24 | 40 | (7)   |                                                        |
|    | L5 | (1)   |                                                        |
| 25 | 10 | 2 F   | $\frac{\xi_i}{16}$ in (6)                              |
|    | 40 | (6)   |                                                        |
| 26 | 50 | (6)   |                                                        |
|    | 7J | (6)   |                                                        |
| 27 | 40 | (6)   | $(\frac{1}{16})^2 [(kb)^2 (\xi_v - 1) - \xi_i^2]$ in A |
|    | L5 | (7)   |                                                        |
| 28 | L0 | (6)   |                                                        |



|    |    |       |                                                    |
|----|----|-------|----------------------------------------------------|
|    | 50 | 28 L  |                                                    |
| 29 | 26 | 149 F | enter R1, leave $\frac{1}{16} X_1'$ in A           |
|    | 50 | (2)   | clear Q                                            |
| 30 | 00 | 2 F   | form $\frac{1}{4} X_1'$                            |
|    | J0 | (2)   | waste                                              |
| 31 | J4 | 4 F   |                                                    |
|    | 50 | 31 L  | print $\frac{1}{4} X_1'$                           |
| 32 | 26 | 59 F  |                                                    |
|    | L5 | (1)   |                                                    |
| 33 | L0 | (9)   | test: $(\frac{\xi_1}{4} - \frac{\xi_0 + .088}{4})$ |
| ←  | 36 | 39 L  |                                                    |
| 34 | L5 | (1)   | advance $\xi_1$                                    |
|    | L4 | (10)  | $\frac{\xi_1 + .01}{4}$ in (1)                     |
| 35 | 40 | (1)   |                                                    |
|    | 50 | 11 L  |                                                    |
| 36 | K5 | F     | advance (500) address                              |
|    | 42 | 11 L  |                                                    |
| 37 | 50 | 12 L  |                                                    |
|    | K5 | F     | advance (510) address                              |
| 38 | 42 | 12 L  |                                                    |
| ←  | 22 | 7 L   | transfer control to 7 L                            |
| 39 | L5 | (11)  |                                                    |
|    | 42 | 11 L  | reset (500)                                        |



|         |      |                   |                                                             |
|---------|------|-------------------|-------------------------------------------------------------|
| 40      | L5   | (12)              | ] reset (510)                                               |
|         | 42   | 12 L              |                                                             |
| 41      | 59   | (2)               | ] test: $\frac{1}{2}$ in A, clear Q<br>$\frac{4.0 - kb}{8}$ |
|         | L0   | 523 F             |                                                             |
| 42      | ← 36 | L                 |                                                             |
|         | 0F   | F                 |                                                             |
| 43 (1)  | 00   | F                 |                                                             |
|         | 00   | F                 |                                                             |
| 44 (2)  | 00   | F                 |                                                             |
|         | 00   | F                 |                                                             |
| 45 (3)  | 40   | F                 |                                                             |
|         | 00   | 140 000 000 000 J |                                                             |
| 46 (4)  | 00   | F                 |                                                             |
|         | 00   | F                 |                                                             |
| 47 (5)  | 00   | F                 |                                                             |
|         | 00   | 250 000 000 000 J |                                                             |
| 48 (6)  | 00   | F                 |                                                             |
|         | 00   | F                 |                                                             |
| 49 (7)  | 00   | F                 |                                                             |
|         | 00   | F                 |                                                             |
| 50 (8)  | 00   | F                 |                                                             |
|         | 00   | 22 000 000 000 J  |                                                             |
| 51 (9)  | 00   | F                 |                                                             |
|         | 00   | F                 |                                                             |
| 52 (10) | 00   | F                 |                                                             |
|         | 00   | 2 500 000 000 J   |                                                             |



|    |      |    |       |
|----|------|----|-------|
| 53 | (11) | 00 | F     |
|    |      | 00 | 500 F |
| 54 | (12) | 00 | F     |
|    |      | 00 | 510 F |

Auxiliary Subroutine

|   |    |       |                                                                         |
|---|----|-------|-------------------------------------------------------------------------|
| 0 | 40 | 18 F  | rescue $X_i/4$                                                          |
|   | K5 | F     |                                                                         |
| 1 | 42 | 8 L   | plant link                                                              |
|   | L5 | 18 F  |                                                                         |
| 2 | 50 | 2 F   | enter V8, leave $J_0(X) \cdot 2^{-19}$ in 6                             |
|   | 50 | 2 L   | and $J_1(X) \cdot 2^{-19}$ in 7.                                        |
| 3 | 26 | 158 F |                                                                         |
|   | L5 | 18 F  |                                                                         |
| 4 | 10 | 2 F   | form $X/16$ , store in 19.                                              |
|   | 40 | 19 F  |                                                                         |
| 5 | 50 | 6 F   |                                                                         |
|   | 79 | 19 F  |                                                                         |
| 6 | 50 | (2)   |                                                                         |
|   | 66 | 7 F   | form $-\frac{1}{16} \times \frac{J_0(X)}{J_1(X)} - \frac{1}{16} \gamma$ |
| 7 | S5 | F     |                                                                         |
|   | L0 | (4)   |                                                                         |
| 8 | J0 | (2)   | waste                                                                   |
|   | 22 | F     |                                                                         |



TABLE B.1

|       |       |         |
|-------|-------|---------|
| .275  |       |         |
| .1375 | .6711 | .6730   |
| .1400 | .6725 | .6725 ← |
| .1425 | .6739 | .6720   |
| .1450 | .6753 | .6715   |
| .1475 | .6767 | .6709   |
| .1500 | .6781 | .6704   |
| .1525 | .6795 | .6698   |
| .1550 | .6809 | .6692   |
| .1575 | .6823 | .6687   |
| .1600 | .6837 | .6681   |
| .325  |       |         |
| .3075 | .7508 | .7514   |
| .3100 | .7517 | .7503 ← |
| .3125 | .7526 | .7493   |
| .3150 | .7535 | .7482   |
| .3175 | .7544 | .7472   |
| .3200 | .7553 | .7461   |
| .3225 | .7562 | .7450   |
| .3250 | .7571 | .7440   |
| .3275 | .7580 | .7429   |
| .3300 | .7588 | .7418   |
| .375  |       |         |
| .4800 | .8013 | .8044   |
| .4825 | .8018 | .8029   |
| .4850 | .8024 | .8014 ← |
| .4875 | .8030 | .7999   |
| .4900 | .8035 | .7984   |
| .4925 | .8041 | .7968   |
| .4950 | .8046 | .7953   |
| .4875 | .8052 | .7937   |
| .5000 | .8057 | .7921   |
| .5025 | .8063 | .7906   |
| .425  |       |         |
| .6525 | .8335 | .8375   |
| .6550 | .8339 | .8355   |
| .6575 | .8343 | .8335 ← |
| .6600 | .8346 | .8316   |
| .6625 | .8350 | .8296   |
| .6650 | .8354 | .8276   |
| .6675 | .8357 | .8256   |
| .6700 | .8361 | .8235   |
| .6725 | .8365 | .8215   |
| .6750 | .8368 | .8194   |



TABLE B.1 (CONTINUED)

|       |       |         |
|-------|-------|---------|
| .475  |       |         |
| .8200 | .8548 | .8576   |
| .8225 | .8551 | .8552   |
| .8250 | .8554 | .8528 ← |
| .8275 | .8556 | .8504   |
| .8300 | .8559 | .8479   |
| .8325 | .8561 | .8455   |
| .8350 | .8564 | .8430   |
| .8375 | .8567 | .8405   |
| .8400 | .8569 | .8380   |
| .8425 | .8572 | .8355   |
| .525  |       |         |
| .9750 | .8692 | .8771   |
| .9775 | .8694 | .8743   |
| .9800 | .8696 | .8715 ← |
| .9825 | .8698 | .8687   |
| .9850 | .8700 | .8658   |
| .9875 | .8702 | .8630   |
| .9900 | .8704 | .8601   |
| .9925 | .8705 | .8572   |
| .9950 | .8707 | .8543   |
| .9975 | .8709 | .8514   |

A parameter tape was placed in the photoelectric tape reader after the main program tape had been read into the computer. The parameter tape listed the Bessel functions and constants which were required for the computations. The fractions had to appear on the tape in the proper sequence, which was: Ten values of  $K_0(\xi)$ , ten corresponding values of  $K_1(\xi)$ , and initial  $J/4$ ,  $X_1/4$ ,  $X_2/4$ , and  $k_0 b/8$ . The total parameter tape was comprised of six sets of 24 fractions each, where 24 fractions were required for each value of  $k_0 b$ . The entries on the tape for  $k_0 b = 3.0$  were as follows:

+12569493  
+12415057  
+12262656  
+12112263  
+11963846  
+11817379



```

+11672831
+11530177
+11389387
+112504361
+15546507
+15341358
+15139211
+14940014
+14743718
+14550275
+14359637
+14171756
+13986588
+138040877
+48
+79
+815
+375 N

```

A decimal point before the first digit is assumed by the ILLIAC. The first ten entries are  $K_0(1.92)$ ,  $K_0(1.93)$ , ...,  $K_0(2.01)$  and the second ten fractions are  $K_1(1.92)$ ,  $K_1(1.93)$ , ...,  $K_1(2.01)$ . The terminating symbol N following  $k_0/8$  tells ILLIAC to stop reading in fractions and control is returned to the main routine.

## B.2

The second program utilized library routine 11 in addition to all of the library routines which were used in the first program. Routine 11 computes an interpolate from a table of function values stored in the computer memory. It uses Neville's method of successive linear interpolations. The user specifies the depth,  $d$ , of interpolation, which implies the highest order "difference" used in the interpolation formula. In this case, the depth was specified as four. The function  $\frac{1}{16} \mathcal{E}_r(\xi) \frac{K_0(\xi)}{K_1(\xi)}$  is fairly linear and more amenable to interpolation than the  $K_n$  Bessel function. Consequently, values of this function were read into the computer for  $\xi$  in the neighborhood of  $\xi_s$  and  $X_s$ , and interpolation was performed



within the sequential table of values. The first program had determined the intersection as occurring between  $\xi_1$  and  $\xi_{i+1}$ , where  $\xi_{i+1} - \xi_1 = .01$ . The second program increased  $\xi$  within this interval in increments of .0001, interpolating to obtain  $\frac{1}{16} \xi_r(\xi) \frac{K_0(\xi)}{K_1(\xi)}$  for each  $\xi$ , until intersection was detected by the method shown in Fig. 21. When intersection was detected, the computer printed out  $\xi$ ,  $X$ , and  $X'$  on each side of crossover in the following form:

$$\begin{array}{ccc} k_o b/8 & & \\ \frac{1}{4} \xi_i & \left| \frac{1}{4} X_i \right| & \left| \frac{1}{4} X'_i \right| \\ \frac{1}{4} \xi_{i+1} & \left| \frac{1}{4} X_{i+1} \right| & \left| \frac{1}{4} X'_{i+1} \right| \end{array}$$

Table B.2 presents the numerical results for each of the six values of  $k_o b$ . Table I was constructed from these data.

TABLE B.2

|        |        |        |
|--------|--------|--------|
| .275   |        |        |
| .14005 | .67251 | .67252 |
| .14007 | .67252 | .67252 |
| .325   |        |        |
| .30820 | .75107 | .75107 |
| .30822 | .75108 | .75106 |
| .375   |        |        |
| .48380 | .80214 | .80215 |
| .48382 | .80214 | .80213 |
| .425   |        |        |
| .65672 | .83415 | .83415 |
| .65675 | .83415 | .83413 |
| .475   |        |        |
| .82260 | .85510 | .85512 |
| .82262 | .85510 | .85510 |
| .525   |        |        |
| .98157 | .86969 | .86972 |
| .98160 | .86969 | .86969 |



The procedure for calculating  $X$  and  $X'$  was similar to that used in the first program. The only essential difference was the addition of routine I1 to perform the interpolation. The tape format and order codes written for the main routine and auxiliary subroutine are as follows:

Tape Format

|               |                                         |
|---------------|-----------------------------------------|
| 00 200 K      |                                         |
| (N12) 00 K    | Routine N12                             |
| (P16) 00 K    | Routine P16                             |
| (R1) 00 K     | Routine R1                              |
| 00 3 K        | } pre-set parameter for H1              |
| 00 F 00 725 F |                                         |
| 00 304 K      | Routine H1                              |
| 00 3 K        | } pre-set parameter for I1              |
| 00 F 00 8 F   |                                         |
| 00 F 00 F     |                                         |
| 00 338 K      | Routine I1                              |
| 00 K          | Routine V8                              |
| 00 K          | Main Routine                            |
| 00 725 K      | Auxiliary Subroutine                    |
| 24 646 N      | Stop - Transfer Control to Main Routine |



Main Routine

|   |      |       |                                                    |
|---|------|-------|----------------------------------------------------|
| 0 | 50   | 21 F  |                                                    |
|   | 50   | L     | read in 12 fractions, locations 21                 |
| 1 | 26   | 200 F | through 32                                         |
|   | J0   | (1)   | waste                                              |
| 2 | 50   | 54 F  |                                                    |
|   | 50   | 2 L   | read 12 fractions into 54 through 65               |
| 3 | 26   | 200 F |                                                    |
|   | ← 26 | 75 L  |                                                    |
| 4 | → 50 | 190 F |                                                    |
|   | 50   | 4 L   | read 6 fractions into 190 through 195              |
| 5 | 26   | 200 F |                                                    |
|   | J0   | (1)   | waste                                              |
| 6 | → 50 | 40 F  |                                                    |
|   | 50   | 6 L   | read <u>parameter constants</u> into 40 through 44 |
| 7 | 26   | 200 F |                                                    |
|   | J0   | (1)   | waste                                              |
| 8 | 52   | 45 F  |                                                    |
|   | 50   | 8 L   | read <u>integer</u> $a \cdot 2^{-39}$ into 45      |
| 9 | 26   | 200 F |                                                    |



|    |    |                  |                                 |
|----|----|------------------|---------------------------------|
|    | L5 | 45 F             |                                 |
| 10 | 42 | 17 L             | insert a into I1 entry          |
|    | 92 | 707 F            | number shift                    |
| 11 | 92 | 131 F            | carriage return and line feed   |
|    | 92 | 515 F            | delay                           |
| 12 | J0 | (1)              | waste                           |
|    | L5 | 43 F             |                                 |
| 13 | J4 | 3 F              | enter P16, print kb/8           |
|    | 50 | 13 L             |                                 |
| 14 | 26 | 239 F            |                                 |
|    | L5 | 40 F             |                                 |
| 15 | 40 | (2)              | puts initial $\xi_t$ into (2)   |
|    | L5 | (2)              | current $\xi_t$ in A            |
| 16 | 50 | 4 F              |                                 |
|    | 50 | 16 L             |                                 |
| 17 | 26 | 338 F            | enter I1, interpolates to leave |
|    | 00 | F                | $\frac{1}{16} \gamma$ in A      |
| 18 | 00 | F                |                                 |
|    | 00 | 10 000 000 000 J |                                 |
| 19 | 40 | (3)              | store interpolate in (3)        |
|    | L5 | 41 F             |                                 |
| 20 | 40 | 10 F             | $X_1/4$ in 10                   |
|    | L5 | 42 F             | $X_2/4$ in 12                   |
| 21 | 40 | 12 F             |                                 |



|    |    |       |                                                        |
|----|----|-------|--------------------------------------------------------|
|    | 50 | 21 L  |                                                        |
| 22 | 26 | 304 F | enter H1, leave root $X_i/4$ in A                      |
|    | 40 | (4)   | store $X_i/4$ in (4)                                   |
| 23 | L5 | (2)   |                                                        |
|    | 10 | 2 F   | forms $\xi_i/4$ from $\xi_t$                           |
| 24 | L4 | 44 F  |                                                        |
|    | 40 | (5)   | store $\xi_i/4$ in (5)                                 |
| 25 | 50 | 43 F  |                                                        |
|    | 7J | 43 F  | $(kb/8)^2$ in (6)                                      |
| 26 | 40 | (6)   |                                                        |
|    | 50 | (6)   |                                                        |
| 27 | 7J | (7)   | $(\frac{1}{16})^2 [(kb)^2 (\xi_r - 1)]$ in (6)         |
|    | 40 | (6)   |                                                        |
| 28 | L5 | (5)   |                                                        |
|    | 10 | 2 F   | $\xi_i/16$ in (8)                                      |
| 29 | 40 | (8)   |                                                        |
|    | 50 | (8)   |                                                        |
| 30 | 7J | (8)   | $(\xi_i/16)^2$ in (8)                                  |
|    | 40 | (8)   |                                                        |
| 31 | 41 | F     | clear A and location 0                                 |
|    | L5 | (6)   |                                                        |
| 32 | L0 | (8)   | $(\frac{1}{16})^2 [(kb)^2 (\xi_r - 1) - \xi_i^2]$ in A |
|    | 50 | 32 L  |                                                        |
| 33 | 26 | 295 F | enter R1, leave $\frac{1}{16} X_i'$ in A               |
|    | 50 | (1)   | clear Q                                                |
| 34 | 00 | 2 F   |                                                        |
|    | 40 | (9)   | $X_i'/4$ in (9)                                        |



|    |      |       |                                     |
|----|------|-------|-------------------------------------|
| 35 | L5   | (4)   | ] test: $X_i/4 - X_i'/4$            |
|    | L0   | (9)   |                                     |
| 36 | ← 32 | 41 L  | ] transfer to print                 |
|    | L5   | (5)   |                                     |
| 37 | 40   | (10)  | ] $\xi_i/4$ in (10)                 |
|    | L5   | (4)   |                                     |
| 38 | 40   | (11)  | ] $X_i/4$ in (11)                   |
|    | L5   | (9)   |                                     |
| 39 | 40   | (12)  | ] $X_i'/4$ in (12)                  |
|    | L5   | (2)   |                                     |
| 40 | L4   | (13)  | ] advance $\xi_t$ by .0001          |
|    | 40   | (2)   |                                     |
| 41 | ← 22 | 15 L  | ] carriage return, line feed, delay |
|    | 92   | 131 F |                                     |
| 42 | 92   | 515 F | ] $\xi_i/4$                         |
|    | L5   | (10)  |                                     |
| 43 | J4   | 5 F   | ] print $\xi_i/4$                   |
|    | 50   | 43 L  |                                     |
| 44 | 26   | 239 F | ] four spaces                       |
|    | 92   | 975 F |                                     |
| 45 | J0   | (1)   | ] waste                             |
|    | L5   | (11)  |                                     |
| 46 | J4   | 5 F   | ] print $X_i/4$                     |
|    | 50   | 46 L  |                                     |
| 47 | 26   | 239 F | ] four spaces                       |
|    | 92   | 975 F |                                     |
| 48 | J0   | (1)   | waste                               |



|      |     |       |                                            |
|------|-----|-------|--------------------------------------------|
|      | L5  | (12)  |                                            |
| 49   | J4  | 5 F   |                                            |
|      | 50  | 49 L  | print $X_i'/4$                             |
| 50   | 26  | 239 F |                                            |
|      | 92  | 131 F | carriage return and line feed              |
| 51   | 92  | 515 F | delay                                      |
|      | L5  | (5)   |                                            |
| 52   | J4  | 5 F   |                                            |
|      | 50  | 52 L  | print $\xi_{i+1}/4$                        |
| 53   | 26  | 239 F |                                            |
|      | 92  | 975 F |                                            |
| 54   | J0  | (1)   | waste                                      |
|      | L5  | (4)   |                                            |
| 55   | J4  | 5 F   |                                            |
|      | 50  | 55 L  | print $X_{i+1}/4$                          |
| 56   | 26  | 239 F |                                            |
|      | 92  | 975 F |                                            |
| 57   | J0  | (1)   |                                            |
|      | L5  | (9)   |                                            |
| 58   | J4  | 5 F   |                                            |
|      | 50  | 58 L  | print $X_{i+1}'/4$                         |
| 59   | 26  | 239 F |                                            |
|      | 59  | (1)   | $\frac{1}{2}$ in A, clear Q                |
| 60   | L0  | 43 F  | test: $\frac{4.0 - kb}{8}$                 |
| ← 36 | 6 L |       | transfer control to read in new parameters |
| 61   | 0F  | F     |                                            |
|      | 00  | F     |                                            |



62 (1) 00 F 00 F  
 63 (2) 00 F 00 F  
 64 (3) 00 F 00 F  
 65 (4) 00 F 00 F  
 66 (5) 00 F 00 F  
 67 (6) 00 F 00 F  
 68 (7) 00 F 00 390 000 000 000 J

69 (8) 00 F 00 F  
 70 (9) 00 F 00 F  
 71 (10) 00 F 00 F  
 72 (11) 00 F 00 F  
 73 (12) 00 F 00 F

74 (13) 00 F 00 100 000 000 J

75 50 91 F  
     50 75 L  
 76 26 200 F  
 ← 26 4 L

read 6 fractions into 91 through 96

#### Auxilliary Subroutine

|   |    |       |                                                                                                      |
|---|----|-------|------------------------------------------------------------------------------------------------------|
| 0 | 40 | 18 F  | rescue $X_1/4$                                                                                       |
|   | K5 | F     |                                                                                                      |
| 1 | 42 | 8 L   | plant link                                                                                           |
|   | L5 | 18 F  | $X/4$ in A                                                                                           |
| 2 | 50 | 2 F   |                                                                                                      |
|   | 50 | 2 L   | enter V8                                                                                             |
| 3 | 26 | 389 F | $\begin{cases} J_0(X) \cdot 2^{-19} & \text{in 6} \\ J_1(X) \cdot 2^{-19} & \text{in 7} \end{cases}$ |



|   |    |      |                                                             |
|---|----|------|-------------------------------------------------------------|
|   | L5 | 18 F |                                                             |
| 4 | 10 | 2 F  | form $X/16$ , store in 19 F                                 |
|   | 40 | 19 F |                                                             |
| 5 | 50 | 6 F  |                                                             |
|   | 79 | 19 F | $-\frac{X}{16} J_0(X) \cdot 2^{-19}$ in A                   |
| 6 | 50 | (1)  | clear Q                                                     |
|   | 66 | 7 F  | $-\frac{X}{16} \frac{J_0(X)}{J_1(X_I)}$ in A                |
| 7 | S5 | F    |                                                             |
|   | L0 | (3)  | $-\frac{X}{16} \frac{J_0(X)}{J_1(X)} - \frac{1}{16} Y$ in A |
| 8 | J0 | (1)  | waste                                                       |
| ← | 22 | F    |                                                             |

The parameter tape which was read into the computer after the main program tape listed the function values,  $\frac{1}{16} \xi_r(\xi) \frac{K_0(\xi)}{K_1(\xi)}$ , required by the interpolation routine. It also included constants telling the computer where the table was stored, the initial value of  $\xi$ ,  $X_1/4$ ,  $X_2/4$ , and  $k_0 b/8$ . The parameter tape as it was read into ILLIAC is as follows:

```

+14217 205
+14363 079
+14509 095
+14655 256
+14801 554
+14947 990
+45745 8333
+45902 4109
+46059 0048
+46215 6141
+46372 2411
+46528 8834N

+05055 518
+05089 839
+05216 642
+05343 995
+05471 883
+05600 291

```



+35300 6528  
+35455 7627  
+35610 9008  
+35766 0675  
+35921 2618  
+36076 4848N

+24685 093  
+24837 399  
+24989 766  
+25142 182  
+25294 660  
+25447 191N

+55638 3502  
+55795 7729  
+55953 2083  
+56110 6549  
+56268 1103  
+56425 5788N

+555+66+68+275+N+N

+23+74+76+325+25N+N

+93+79+81+375+25N+N

+62+82+84+425+5N+N

+285+84+86+475+75N+N

+92+86+88+525+75N+100N



## APPENDIX C

The integral (6.6) which was needed in order to compute the excitation efficiency is

$$N^R = \frac{1}{\pi/2} \int_0^{\frac{88}{90} \pi/2} \cos \theta \left| F(\theta) \right|^2 d\theta \quad (6.6)$$

where

$$F(\theta) = \frac{J_1(k_0 a w)}{w J_0(k_0 b w) H_1^{(1)}(k_0 b \cos \theta) - \epsilon_r \cos \theta J_1(k_0 b w) H_0^{(1)}(k_0 b \cos \theta)}$$

and

$$w = \sqrt{\epsilon_r - \sin^2 \theta}$$

The program which was written to determine  $N^R$  utilized ILLIAC library routine E2, which computes an approximation to the integral

$$\frac{1}{b-a} \int_a^b f(x) dx \text{ by Simpson's rule. This routine requires that the}$$

user supply tabulated values of the integrand at an even number of equally spaced intervals. The range of integration, which was zero to 88 degrees, was subdivided into 44 intervals of two degrees each.

For any  $k_0 a$  and  $k_0 b$ , the integrand was computed at the 45 points  $0^\circ, 2^\circ, 4^\circ, 6^\circ, \dots, 86^\circ, 88^\circ$ . Since the integrand contains  $F(\theta)$ ,

which involves a division, it was necessary to scale the numerator

whenever it exceeded the denominator in order to maintain  $|F(\theta)| < 1$ ;

otherwise the capacity of the ILLIAC registers would be exceeded. This



was done by multiplying the numerator by  $1/2$  as often as needed until all 45 values of  $F(\theta)$  were within range. Consequently, the computed integral was scaled by an integer number of multiplications by the factor  $1/2$ . The scaled integral,  $I$ , which was computed was equal to

$$I = \frac{1}{\frac{88}{90} \frac{\pi}{2}} \int_0^{\frac{88}{90} \frac{\pi}{2}} \left(\frac{1}{2}\right)^{p+1} \cos \theta |F(\theta)|^2 d\theta \quad (C.1)$$

where  $p$  is the integer scale factor.

It follows from (6.6) and (C.1) that  $N^R$  is obtained from  $I$  by

$$N^R = \frac{88}{90} \frac{1}{2^{p+1}} I \quad (C.2)$$

For a particular choice of  $k_0 b$ , the integral (C.1) was calculated for  $k_0 a$  equal to 0.2, 0.4, 0.6, ... until  $k_0 a$  took on the value  $k_0 b$ ; then  $k_0 b$  was changed to a new value, and the process was repeated. The scaled quantities  $k_0 b/8$  and  $k_0 a/8$  were used in the computations. The results were printed in four columns which listed, respectively,  $k_0 b/8$ ,  $k_0 a/8$ ,  $p$ , and  $I$ . Relation (C.2) was then used to compute the corresponding value of  $N^R$ . Table C1 presents the results as printed out by ILLIAC for  $k_0 b/8$  equal to 0.425, that is,  $k_0 b$  equal to 3.4. The program which was written to compute the integral (C.1) was comprised of 143 order pairs. Due to its length, the order code will not be given in detail. However, it should be mentioned that library routines V8 (Bessel functions), T5 (sine-cosine), R1 (square root), P16 (print routine), and E2 (integration) were utilized by the main program.



TABLE C1.

| $k_{\text{O}} b/8$ | $k_{\text{O}} a/8$ | $p$ | $I$   |
|--------------------|--------------------|-----|-------|
| .425               | .025               | 1   | .0304 |
| .425               | .050               | 1   | .1136 |
| .425               | .075               | 1   | .2280 |
| .425               | .100               | 2   | .1722 |
| .425               | .125               | 2   | .2167 |
| .425               | .150               | 3   | .1184 |
| .425               | .175               | 3   | .1142 |
| .425               | .200               | 3   | .0974 |
| .425               | .225               | 2   | .1450 |
| .425               | .250               | 2   | .0913 |
| .425               | .275               | 1   | .0914 |
| .425               | .300               | 1   | .0334 |
| .425               | .325               | 1   | .0149 |
| .425               | .350               | 1   | .0310 |
| .425               | .375               | 1   | .0682 |
| .425               | .400               | 1   | .1097 |
| .425               | .425               | 1   | .1403 |



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